A Proofs of Linear Case

Throughout the appendix, for ease of notation, we overload the definition of the function $d_{TV}(\cdot,\cdot)$. When inputs are random variables, it represent the TV distance between the distributions of those random variables.

Lemma 4.2. *Let $\{(x_i,y_i)\}_{i=1}^n$ be i.i.d. random variables such that $y_i = x_i \cdot w^* + \mathcal{N}(0,1)$. Then, for $n \geq \frac{k}{2}$, with probability $1 - e^{-\Omega(n)}$, the MLE $\widehat{w}$ satisfies*

$$\widetilde{d}(\widehat{w}, w^*) \leq \sqrt{\frac{k}{2n}}.$$

The proof of this lemma requires Lemma A.1, which characterizes the distribution of the residual error of the MLE.

Lemma A.1. *Given $y \in \mathbb{R}^n, X \in \mathbb{R}^{n\times k}$ satisfying $y = Xw^* + \eta$, where $\eta \sim \mathcal{N}(0, \sigma^2 I_n)$, the least square solution $\widehat{w}$ satisfies*

$$Xw^* - X\widehat{w} \sim \mathcal{N}(0, \sigma^2 X(X^TX)^{-1}X^T) \Rightarrow \mathbb{E}[\|X\hat{w} - Xw^*\|^2] = \sigma^2 k.$$

Proof. The least squares solution is given by

$$\begin{aligned}\hat{w} &= (X^TX)^{-1}X^Ty,\\ &= (X^TX)^{-1}X^T(Xw^* + \eta),\\ &= w^* + (X^TX)^{-1}X^T\eta.\end{aligned}$$

Multiplying on the left by X, we have

$$X\hat{w} = Xw^* + X(X^TX)^{-1}X^T\eta.$$

Since η is i.i.d. Gaussian with variance σ^2, we have,

$$\begin{aligned}X(X^TX)^{-1}X^T\eta &\sim \mathcal{N}(0, \sigma^2 X(X^TX)^{-1}X^TX(X^TX)^{-1}X^T)\\ &\sim \mathcal{N}(0, \sigma^2 X(X^TX)^{-1}X^T)\end{aligned}$$

This implies

$$\begin{aligned}\mathbb{E}[\|X\hat{w} - Xw^*\|^2] &= \sigma^2 \mathrm{Tr}[X(X^TX)^{-1}X^T],\\ &= \sigma^2 \mathrm{Tr}[(X^TX)^{-1}X^TX],\\ &= \sigma^2 k.\end{aligned}$$

□

Proof of Lemma 4.2. The KL divergence between two Gaussians $P = \mathcal{N}(\mu_1, \Sigma)$ and $Q = \mathcal{N}(\mu_2, \Sigma)$ is:

$$d_{KL}(P\|Q) = \frac{1}{2}(\mu_1 - \mu_2)\Sigma^{-1}(\mu_1 - \mu_2).$$

By Pinsker's inequality, this implies

$$d_{TV}(P\|Q) \leq \min\left\{1, \frac{1}{2}\sqrt{(\mu_1 - \mu_2)\Sigma^{-1}(\mu_1 - \mu_2)}\right\}.$$

Hence, the empirical TV on the dataset can be bounded by

$$\begin{aligned}
\frac{1}{n}\sum_i d_{TV}(p_{\widehat{w}}(y|x_i), p_{w^*}(y|x_i)) &\leq \frac{1}{n}\sum_i \min\left\{1, \frac{1}{2}\frac{\left|x_i^T(\widehat{w}-w^*)\right|}{\sigma}\right\}, \\
&\leq \sqrt{\frac{1}{n}\sum_i \min\left\{1, \frac{1}{2}\frac{\left|x_i^T(\widehat{w}-w^*)\right|}{\sigma}\right\}^2}, \\
&\leq \sqrt{\min\left\{1, \frac{1}{4n}\sum_i \frac{\left(x_i^T(\widehat{w}-w^*)\right)^2}{\sigma^2}\right\}}, \\
&= \sqrt{\min\left\{1, \frac{1}{4n}\frac{1}{\sigma^2}\|X(\widehat{w}-w^*)\|^2\right\}}.
\end{aligned}$$

where the second line follows from Jensen's inequality.

By Lemma A.1, we have

$$\mathbb{E}[\|X(\widehat{w}-w^*)\|^2] = \sigma^2 k.$$

which implies that with probability $1-e^{-\Omega(n)}$, we have

$$\|X(\widehat{w}-w^*)\|^2 \leq 2\sigma^2 k.$$

Substituting in the earlier inequality, we get

$$\frac{1}{n}\sum_i d_{TV}(p_{\widehat{w}}(y|x_i), p_{w^*}(y|x_i)) \leq \sqrt{\min\left\{1, \frac{k}{2n}\right\}} = \sqrt{\frac{k}{2n}} \text{ for } n \geq \frac{k}{2}.$$

□

Lemma 4.3. *Let $\{x_i\}_{i=1}^n$ be i.i.d. random variables such that $x_i \sim \mathcal{D}_x$. For a sufficiently large constant $C > 0$, and for $n = C\frac{k}{\varepsilon^2}\log\frac{1}{\varepsilon}$ with $n \geq \frac{k}{2}$, we have:*

$$\Pr_{x_i\sim\mathcal{D}_x}\left[\sup_{w\in\mathbb{R}^k}\left|\widetilde{d}(w,w^*) - d(w,w^*)\right| > \varepsilon\right] \leq e^{-\Omega(n\varepsilon^2)}.$$

Proof. The proof is inspired by Theorem 11.2 in [20], with modifications to our setting.

Let Since $f_w(x)$ is bounded, for any fixed w, the Chernoff bound gives

$$\Pr\left[\left|\widetilde{d}(w,w^*) - d(w,w^*)\right| > \alpha\right] \leq e^{-2n\alpha^2}. \tag{11}$$

for any $\alpha > 0$. The challenge lies in constructing a "net" to be able to union bound over $\mathbb{R}^k$ without assuming any bound on w or the covariate x. A net is a partitioning of an space, where within each part, points are close together in some way. In this case, we construct a net using what we will refer to as "ghost" samples.

Ghost samples. First, we construct a "ghost" dataset D'_x consisting of n new samples, drawn i.i.d. $\{x'_i\}_{i\in[n]}$ of $\mathcal{D}_x$. This gives another metric $\widetilde{d}'(\cdot,\cdot)$. Instead of directly considering the distance between $\widetilde{d}(w,w^*)$ and $d(w,w^*)$, it is sufficient to consider the difference between $\widetilde{d}(w,w^*)$ and $\widetilde{d}'(w,w^*)$ i.e.,

$$\Pr\left[\sup_w\left|d(w,w^*) - \widetilde{d}(w,w^*)\right| > \varepsilon\right] \leq 2\Pr\left[\sup_w\left|\widetilde{d}(w,w^*) - \widetilde{d}'(w,w^*)\right| > \varepsilon/2\right]. \tag{12}$$

To see this, let $\bar{w}$ maximize $\widetilde{d}(w,w^*) - \widetilde{d}'(w,w^*)$. Since $\bar{w}$ and $\{x'_i\}_{i\in[n]}$ are independent, by the Chernoff bound,

$$\Pr\left[\left|\widetilde{d}'(\bar{w},w^*) - d(\bar{w},w^*)\right| > \varepsilon/2|D_x\right] \leq e^{-n\varepsilon^2/2} \leq 1/2.$$

for any $(D_x, \bar{w})$ and large enough n. Thus,

$$\begin{aligned}\Pr\left[\left|\widetilde{d}'(\bar{w}, w^*) - \widetilde{d}(\bar{w}, w^*)\right| > \varepsilon/2\right] &\geq \Pr\left[\left|d(\bar{w}, w^*) - \widetilde{d}(\bar{w}, w^*)\right| > \varepsilon \cap \left|d(\bar{w}, w^*) - \widetilde{d}'(\bar{w}, w^*)\right| < \varepsilon/2\right] \\ &= \mathop{\mathbb{E}}_{D_x}\left[1_{\{\left|d(\bar{w},w^*) - \widetilde{d}(\bar{w},w^*)\right| > \varepsilon\}} \Pr\left[\left|d(\bar{w}, w^*) - \widetilde{d}'(\bar{w}, w^*)\right| < \varepsilon/2 | D_x\right]\right] \\ &\geq (1 - 1/2) \Pr\left[\left|d(w, w^*) - \widetilde{d}(w, w^*)\right| > \varepsilon\right],\end{aligned}$$

which implies (12).

Symmetrization. Since D_x and D'_x each have n independent samples, we could instead draw the datasets by first sampling $2n$ elements $x_1, \dots, x_{2n}$ from $\mathcal{D}_x$, then randomly partition this sample into two equal datasets. Let $s_i \in \{\pm 1\}$ so $s_i = 1$ if z_i lies in D'_x and -1 if it lies in D_x. Then

$$\widetilde{d}'(\bar{w}, w^*) - \widetilde{d}(\bar{w}, w^*) = \frac{1}{n} \sum_{i=1}^{2n} s_i \cdot d_{TV}(p_w(y|x_i), p_{w^*}(y|x_i)).$$

For a fixed w and $x_1, \dots, x_{2n}$, the random variables $(s_1, \dots, s_{2n})$ are a permutation distribution, so negatively associated. Then the variables $s_i \cdot d_{TV}(p_w(y|x_i), p_{w^*}(y|x_i))$ are monotone functions of s_i, so also negatively associated. They are also bounded in $[-1, 1]$. Hence we can apply a Chernoff bound:

$$\Pr\left[\left|\widetilde{d}'(\bar{w}, w^*) - \widetilde{d}(\bar{w}, w^*)\right| > \varepsilon\right] < e^{-n\varepsilon^2/2} \tag{13}$$

for any fixed w.

Constructing a net. We partition $\mathbb{R}^k$ the space of w s.t. if w, w' are in the same partition then,

$$\left|d_{TV}(p_w(y|x), p_{w^*}(y|x)) - d_{TV}(p_{w'}(y|x), p_{w^*}(y|x))\right| < \alpha.$$

for each x in the dataset $x_1, \dots, x_{2n}$. Then take the intersection of all $2n$ partitions to construct a net over $\mathbb{R}^k$.

As the total variation distance is a unimodal function of $x_i \cdot w - x_i \cdot w^*$, we partition w the sets

$$\{w : d_{TV}(p_w(y|x_i), p_{w^*}(y|x_i)) \in [j\alpha, (j+1)\alpha]$$

where j goes from 0 to $1/\alpha - 1$. So the space of w, $\mathbb{R}^k$ is partitioned by $2n$ sets of $1/\alpha$ parallel hyper-planes. Then the total number of cells is at most

$$\sum_{i=0}^{k} \binom{2n}{i} (2/\alpha)^i \leq 2\left(\frac{4en}{\alpha k}\right)^k$$

We define a net N by choosing one representative of each cell in the partition, so $|N| \leq e^{2k \log \frac{n}{\alpha k}}$. By (13),

$$\Pr\left[\max_{w \in N}\left|\widetilde{d}'(\bar{w}, w^*) - \widetilde{d}(\bar{w}, w^*)\right| > \varepsilon\right] < |N| e^{-n\varepsilon^2/2} \leq e^{2k \log \frac{n}{\alpha k} - \varepsilon^2 n/2}.$$

Finally, for any $w \in \mathbb{R}^d$ let $\bar{w} \in N$ be the representative of its cell. By definition of the cells,

$$|d_{TV}(p_w(y|x_i), p_{w^*}(y|x_i)) - d_{TV}(p_{\bar{w}}(y|x_i), p_{w^*}(y|x_i))| < \alpha$$

for all $i \in [2n]$. Thus

$$\left|\left(\widetilde{d}'(w, w^*) - \widetilde{d}(w, w^*)\right) - \left(\widetilde{d}'(\bar{w}, w^*) - \widetilde{d}(\bar{w}, w^*)\right|\right| \leq \left|\widetilde{d}(w, w^*) - \widetilde{d}(\bar{w}, w^*)\right| + \left|\widetilde{d}'(w, w^*) - \widetilde{d}'(\bar{w}, w^*)\right| \leq 2\alpha$$

and so

$$\Pr\left[\sup_{w \in \mathbb{R}^d}\left|\widetilde{d}'(w, w^*) - \widetilde{d}(w, w^*)\right| > \varepsilon\right] \leq \Pr\left[\max_{w \in N}\left|\widetilde{d}'(w, w^*) - \widetilde{d}(w, w^*)\right| > \varepsilon - 2\alpha\right] \leq e^{2k \log \frac{n}{\alpha k} - (\varepsilon - 2\alpha)^2 n/2}$$

Setting $\alpha = \varepsilon/4$, we have that

$$n \lesssim \frac{1}{\varepsilon^2} k \log \frac{1}{\varepsilon}$$

suffices for

$$\Pr\left[\max_{w \in \mathbb{R}^k} \widetilde{d}'(w, w^*) - \widetilde{d}(w, w^*) > \varepsilon\right] < e^{-\Omega(\varepsilon^2 n)}.$$

□

B ReLU Activation with Scalar y

In this section, we consider the model of

$$y = \phi(w^* \cdot x + \eta), \quad \eta \sim \mathcal{N}(0, 1),$$

where $w^*, x \in \mathbb{R}^k$, $y, \eta \in \mathbb{R}$. We are given samples $(x, y) \in \mathbb{R}^k \times \mathbb{R}$, and want to estimate a $\widehat{w}$ that estimates the distribution of y in TV.

The most challenging aspect of the ReLU setting is that we do not have an expression for the TV suffered by the MLE, such as Lemma 4.2 in the linear case. This forces us to directly analyze the log-likelihood.

For a fixed x, w,, the expectation of the log-likelihood ratio over y is

$$\mathbb{E}_{y}\left[\log \frac{p_w(y \mid x)}{p_{w^*}(y \mid x)}\right] = -d_{KL}(w^* \| w) \leq -2d_{TV}^2(w^*, w),$$

where the last inequality is via Pinsker's inequality. This equation implies that if w is ε-far from w^*, then the expected log-likelihood ratio(LLR) is $< -2\varepsilon^2$. By definition, the MLE has a non-negative LLR. Hence, if the empirical LLR is close to the expectation, this would imply that the MLE has small TV.

However, we only receive a single sample of y per x. For a fixed w, we can prove a Bernstein inequality, showing that given $1/\varepsilon^2 \log(1/\delta)$ samples, the empirical LLR is $< -\varepsilon^2$ for w that are ε-far.

Lemma B.1. *Let $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ be distributions with $\mathbb{E}_i[d_{TV}(p_i, q_i)] \geq \varepsilon$, where we use the uniform measure on $i \in [n]$. Let $x_i \sim p_i$ for $i \in [n]$. Then w.p. $1 - \delta$, $\mathbb{E}_i[\log \frac{q_i(x_i)}{p_i(x_i)}] \leq -\frac{\varepsilon^2}{4}$ for $n \geq O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}\right)$.*

The proof of this Lemma, as well as other Lemmas in this section, can be found in Appendix B.1.

In order to extend this to *all* $w \in \mathbb{R}^k$ that are ε-far, we will construct a cover over $\mathbb{R}^k$ depending on the values the log-likelihood ratio can take, and then apply the Bernstein inequality to each element in the cover.

In order to construct the cover, we first show that the log-likelihood ratio is bounded above by the magnitude of noise in y. For ease of notation, for a fixed $x \in \mathbb{R}^k$, and each $w \in \mathbb{R}^k$, define

$$\theta = \langle x, w \rangle \in \mathbb{R},$$

and let $\theta^* = \langle x, w^* \rangle$.Similar to the notation for w, for each $\theta \in \mathbb{R}$, define p_θ as the distribution of $\phi(\theta + \eta)$ for $\eta \sim N(0, 1)$. Define the log likelihood ratio

$$\gamma_\theta(y) := \log \frac{p_\theta(y|x)}{p_{\theta^*}(y|x)}.$$

The following Lemma states that for a fixed datapoint (x, y), the log-likelihood ratio is bounded by the noise in y:

Lemma B.2. *For any $y = \phi(\theta + \eta)$,*

$$\gamma_\theta(y) = \begin{cases} \log \Phi(-\theta) - \log \Phi(-\theta^*) & \text{if } y = 0 \\ \eta(\theta - \theta^*) - \frac{(\theta - \theta^*)^2}{2} & \text{if } y > 0 \end{cases}$$

and therefore, for all y,

$$\gamma_\theta(y) \leq |\eta|^2/2.$$

Now, as γ is bounded above by $\frac{|\eta|^2}{2}$, and it is concave wrt θ, the following Lemma shows that we can partition θ into $O\left(\frac{A}{\varepsilon}\right)$ intervals, such that in each interval, γ changes by atmost ε, or is very negative, i.e., $\gamma < -A$.

Lemma B.3 (One-dimensional net)**.** *Let $A > B^2 > 1$. There exists a partition of $\mathbb{R}$ into $O(A/\varepsilon)$ intervals such that, for each interval I in the partition and every $y = \phi(\theta^* + \eta)$ with $|\eta| \leq B$, one of the following holds:*

- *For all $\theta \in I$, $\gamma_\theta(y) \leq -A$*
- *For all $\theta, \theta' \in I$, $|\gamma_\theta(y) - \gamma_{\theta'}(y)| \leq \varepsilon$*

Using Lemma B.2 and Lemma B.3, we can form a uniform bound, such that all w that are ε-far from w^* in distribution will have log-likelihood ratio smaller than $-\frac{\varepsilon^2}{4}$ on the training set. With some additional arguments, we can now show that as the MLE has positive log-likelihood ratio, it has small empirical TV.

Lemma B.4. *Let $x_1, \dots, x_n$ be fixed, and $y_i \sim \phi(x_i^T w^* + \eta_i)$ for $\eta_i \sim \mathcal{N}(0,1)$. For $n \geq \frac{1}{\varepsilon^2} k \log \frac{1}{\varepsilon}$, the MLE $\widehat{w}$ satisfies*

$$\widetilde{d}(\widehat{w}, w^*) \leq \varepsilon.$$

This sample complexity guarantees that the MLE is good for the set of empirical $x_i \sim \mathcal{D}_x$, and we need to extend this to the expectation over $x \sim \mathcal{D}_x$, for which we use a Lemma similar to Lemma 4.3 in the linear case, and this completes the proof.

A straight forward combination of Lemma 4.3 and Lemma B.4 gives the following Theorem.

Theorem B.5. *Let $y = \phi\big(x^T w^* + \eta\big)$, for $w^* \in \mathbb{R}^k$, $x \sim \mathcal{D}_x$, and $\eta \sim \mathcal{N}(0,1)$. Then for a sufficiently large constant $C > 0$,*

$$n = C \cdot \frac{k}{\varepsilon^2} \log \frac{1}{\varepsilon}$$

samples of $\{(y_i, x_i)\}_{i=1}^n$ suffices to guarantee that the MLE $\widehat{w}$ satisfies

$$d(\widehat{w}, w^*) \leq \varepsilon.$$

B.1 Proofs

Lemma B.1. *Let $p_1, \dots, p_n$ and $q_1, \dots, q_n$ be distributions with $\mathbb{E}_i[d_{TV}(p_i, q_i)] \geq \varepsilon$, where we use the uniform measure on $i \in [n]$. Let $x_i \sim p_i$ for $i \in [n]$. Then w.p. $1-\delta$, $\mathbb{E}_i[\log \frac{q_i(x_i)}{p_i(x_i)}] \leq -\frac{\varepsilon^2}{4}$ for $n \geq O\big(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}\big)$.*

Proof. Define $\gamma_i(x) = \log \frac{q_i(x)}{p_i(x)}$ and $a_i(x) := \max(\gamma_i(x), -2)$. We have that

$$\mathop{\mathbb{E}}_{i,x}[\gamma_i(x)] = -\mathop{\mathbb{E}}_{i}[d_{KL}(p_i, q_i)] \leq -\mathop{\mathbb{E}}_{i}[2 d_{TV}(p_i, q_i)^2] \leq -2\varepsilon^2$$

and want to show that $\mathbb{E}_i[\gamma_i(x)] \leq -\varepsilon^2/4$ with high probability. Note that $a_i(x) \geq \gamma_i(x)$, so it suffices to show $\mathbb{E}_i[a_i(x)] \leq -\varepsilon^2/4$. We will do this with Bernstein's inequality, for which we need bounds on the moments of $a_i(x)$.

To simplify notation, fix a particular i and consider $p = p_i, q = q_i, a = a_i$, and $x \sim p$.

For a random variable v, define v_+, v_- to be the positive/negative parts of v, respectively, so $v = v_- + v_+$. Define $\Delta(x) = \frac{q(x)}{p(x)} - 1$. We have that $\mathbb{E}_{x\sim p}[\Delta(x)] = 0$, and

$$\mathop{\mathbb{E}}_{x\sim p}[\Delta_+(x)] = \mathop{\mathbb{E}}_{x\sim p}[-\Delta_-(x)] = d_{TV}(p, q). \tag{14}$$

Now, consider the function $b(z) := \max(\log(1+z), -2) - z$. This function is nonpositive over $z \geq -1$, and $b(z) \leq -z^2/2$ for $z \leq 0$. Since

$$a(x) = b(\Delta(x)) + \Delta(x)$$

and $\mathbb{E}_{x\sim p}[\Delta(x)] = 0$, $\mathbb{E}_{x\sim p}[-a(x)] = \mathbb{E}_{x\sim p}[-b(\Delta(x))]$. This means

$$\begin{aligned}\mathop{\mathbb{E}}_{x\sim p}[-a(x)] &= \mathop{\mathbb{E}}_{x\sim p}[-b(\Delta(x))] \geq \mathop{\mathbb{E}}_{x\sim p}[-b(\Delta(x)) 1_{\Delta(x)<0}] \\ &\geq \mathop{\mathbb{E}}_{x\sim p}[\Delta_-^2(x)/2]\end{aligned}$$

or by (14),

$$\mathop{\mathbb{E}}_{x}[-a(x)] \geq \mathop{\mathbb{E}}_{x}[\Delta_-^2(x)/2] \geq \frac{1}{2} d_{TV}(p, q)^2. \tag{15}$$

Bounding the positive higher moments. We have that $p(x)e^{a(x)} = \max(q(x), e^{-2}p(x))$ so

$$\begin{aligned}\mathbb{E}[e^{a(x)}] &= \int \max(q(x), e^{-2}p(x))dx \\ &\leq 1 + e^{-2}\Pr[a(x) = -2].\end{aligned}$$

In the following, we use that $e^t \geq 1 + t$ for all t, as well as $e^t = 1 + t + \sum_{k=2}^{\infty} \frac{1}{k!}t^k$. Therefore

$$\begin{aligned}&1 + e^{-2}\Pr[a(x) = -2] \\ &\geq \mathbb{E}[e^{a(x)}] = \mathbb{E}[e^{a_-(x)}1_{a(x)\leq 0} + e^{a_+(x)}1_{a(x)>0}] \\ &\qquad\geq \mathbb{E}[1 + (a_-(x) + a_+(x)) + \sum_{k=2}^{\infty}\frac{1}{k!}a_+^k(x)] \\ &\qquad= 1 + \mathbb{E}[a(x)] + \sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}[a_+^k(x)]\end{aligned}$$

so

$$\sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}[a_+^k(x)] \leq \mathbb{E}[-a(x)] + e^{-2}\Pr[a(x) = -2].$$

We now show that the $\Pr[a(x) = -2]$ is smaller than the $\mathbb{E}[-a(x)]$ term, by relating to $-b$. When $a(x) = -2$, $\Delta(x) \leq -1+1/e^2$, and $b(\Delta(x)) = -2-\Delta(x) \leq -1$. Since $-b(\Delta(x))$ is non-negative, and at least 1 whenever $a(x) = -2$,

$$\mathbb{E}[-a(x)] = \mathbb{E}[-b(\Delta(x))] \geq \Pr[a(x) = -2] \cdot 1$$

and hence

$$\sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}[a_+^k(x)] \leq (1 + \frac{1}{e^2})\,\mathbb{E}[-a(x)]. \tag{16}$$

In particular, $\mathbb{E}[a_+^k(x)] \leq 2k!\,\mathbb{E}[-a(x)]$ for all $k \geq 2$.

Bounding the second moment of a. We have that

$$\mathbb{E}[a(x)^2] = \mathbb{E}[a_+^2(x) + a_-^2(x)]$$

and $\mathbb{E}[a_+^2(x)] \leq 4\,\mathbb{E}[-a(x)]$ by (16). We now bound $\mathbb{E}[a_-^2(x)]$. Note that $|a_-(x)| \leq \frac{2}{1-1/e^2}|\Delta_-(x)|$ by the construction of a. Therefore

$$a_-^2(x) \leq 6\Delta_-^2(x)$$

and so by (15),

$$\mathbb{E}[a_-^2(x)] \leq 6\,\mathbb{E}[\Delta_-^2(x)] \leq 12\,\mathbb{E}[-a(x)].$$

Thus

$$\mathbb{E}[a^2(x)] \leq 16\,\mathbb{E}[-a(x)]. \tag{17}$$

Bernstein Concentration. Now we can apply Bernstein's inequality (Theorem 2.10 of [8]).

We apply the theorem to $X_i := a_i(x_i)$, which are independent. The theorem uses that

$$\sum_{i=1}^{n}\mathbb{E}[X_i^2] = n\,\mathop{\mathbb{E}}_{i,x}[a_i(x)^2] \leq 16n\,\mathop{\mathbb{E}}_{i,x}[-a_i(x)] =: v$$

by (17), and since

$$\sum_{i=1}^{n}\mathbb{E}[(X_i)_+^k] = n\,\mathop{\mathbb{E}}_{i,x}[a_{i,+}(x)^k] \leq 2k!\,\mathop{\mathbb{E}}_{i,x}[-a_i(x)] \leq \frac{1}{2}vk!$$

so we can set $c = 1$. Applying the theorem, we have that $S = \sum a_i(x_i) - \mathbb{E}[a_i(x_i)]$ satisfies

$$S \leq \sqrt{2v\log\frac{1}{\delta}} + \log\frac{1}{\delta}$$

with probability $1-\delta$. Plugging in v and rescaling by n, with probability $1-\delta$ we have:

$$\mathop{\mathbb{E}}_i[a_i(x_i)] \leq \mathop{\mathbb{E}}_{i,x}[a_i(x)] + O(1) \cdot \sqrt{\mathbb{E}[-a(x)]\frac{1}{n}\log\frac{1}{\delta} + \frac{1}{n}\log\frac{1}{\delta}}$$

By our assumption on n for a sufficiently large constant in the big O, this implies

$$\mathop{\mathbb{E}}_i[a_i(x_i)] \leq \mathop{\mathbb{E}}_{i,x}[a_i(x)] + \frac{1}{6}\varepsilon\sqrt{\mathop{\mathbb{E}}_{i,x}[-a_i(x)]} + \varepsilon^2/8$$

Since by (15), $\varepsilon \leq \sqrt{\mathbb{E}_i[d_{TV}(p_i,q_i)^2]} \leq \sqrt{\mathbb{E}_{i,x}[-2a_i(x)]}$, this means

$$\begin{aligned}
\mathop{\mathbb{E}}_i[\gamma_i(x_i)] \leq \mathop{\mathbb{E}}_i[a_i(x_i)] &\leq (-1+\frac{\sqrt{2}}{6})\mathop{\mathbb{E}}_{i,x}[-a_i(x)] + \varepsilon^2/8 \\
&\leq (-1+\frac{\sqrt{2}}{6})\frac{1}{2}\varepsilon^2 + \frac{1}{8}\varepsilon^2 \\
&\leq -\frac{1}{4}\varepsilon^2
\end{aligned}$$

as desired. $\square$

Lemma B.2. *For any $y = \phi(\theta+\eta)$,*

$$\gamma_\theta(y) = \begin{cases} \log\Phi(-\theta) - \log\Phi(-\theta^*) & \text{if } y = 0 \\ \eta(\theta-\theta^*) - \frac{(\theta-\theta^*)^2}{2} & \text{if } y > 0 \end{cases}$$

and therefore, for all y,

$$\gamma_\theta(y) \leq |\eta|^2/2.$$

Proof. Let $\Phi(x)$ be the cdf of a standard Gaussian. For $y > 0$,

$$\begin{aligned}
\gamma_\theta(y) &= \frac{1}{2}((y-\theta^*)^2 - (y-\theta)^2) \\
&= \frac{1}{2}(\eta^2 - (\eta+\theta^*-\theta)^2) \\
&= \eta(\theta-\theta^*) - \frac{(\theta-\theta^*)^2}{2}
\end{aligned}$$

Thus:

$$\gamma_\theta(y) = \begin{cases} \log\Phi(-\theta) - \log\Phi(-\theta^*) & \text{if } y = 0 \\ \eta(\theta-\theta^*) - \frac{(\theta-\theta^*)^2}{2} & \text{if } y > 0 \end{cases}$$

Now suppose $|\eta| \leq B$. We can upper bound $\gamma_\theta(y)$ for all θ:

- If $y = 0$, then $-\theta^* \geq -B$, so
$$\gamma_\theta(0) \leq -\log\Phi(-\theta^*) \leq -\log e^{-B^2/2} = B^2/2.$$
- If $y > 0$, then
$$\gamma_\theta(y) = (\theta-\theta^*)\eta - \frac{(\theta-\theta^*)^2}{2} \leq \eta^2/2 \leq B^2/2.$$

as desired. $\square$

Lemma B.3 (One-dimensional net)**.** *Let $A > B^2 > 1$. There exists a partition of $\mathbb{R}$ into $O(A/\varepsilon)$ intervals such that, for each interval I in the partition and every $y = \phi(\theta^*+\eta)$ with $|\eta| \leq B$, one of the following holds:*

- *For all $\theta \in I$, $\gamma_\theta(y) \leq -A$*
- *For all $\theta, \theta' \in I$, $|\gamma_\theta(y) - \gamma_{\theta'}(y)| \leq \varepsilon$*

Proof. To define our partition, we actually define two partitions, depending on whether $y = 0$, then intersect them for our final partition.

First, consider $y = 0$. By Lemma B.2, $\gamma_\theta(0)$ is monotonically decreasing in θ, from its maximum of at most $B^2/2$. We can thus define a partition P_1 consisting of intervals of the form $I_i := \{\theta \mid \gamma_\theta(0) \in (B^2/2 - (i+1)\varepsilon, B^2/2 - i\varepsilon)\}$, for $i \in \{0, 1, \dots, (A + B^2/2)/\varepsilon\}$, plus a special interval I' of $\{\theta \mid \gamma_\theta(0) < -A\}$. When $y = 0$, this partition satisfies the desired conclusion to the lemma: $|\gamma_\theta(0) - \gamma_{\theta'}(0)| \leq \varepsilon$ for all $\theta, \theta' \in I_i$, while $\gamma_\theta(0) < -A$ for $\theta \in I'$. Call this partition P_0, which has size $O(A/\varepsilon)$.

Second, consider $y > 0$. Define $R = \sqrt{2A} + B$. Note that $R^2 \lesssim A$ and $(R - B)^2 \geq 2A$. Therefore for $|\theta - \theta^*| \geq R$,

$$\gamma_\theta(y) \leq -\frac{1}{2} \max(0, |\theta - \theta^*| - \eta)^2 \leq -A.$$

Consider any $\theta, \theta' \in [\theta^* - R, \theta^* + R]$ with $\alpha := |\theta - \theta'|$. We have

$$\begin{aligned} |\gamma_\theta(y) - \gamma_{\theta'}(y)| &\leq |\eta(\theta - \theta')| + \frac{1}{2}\left|(\theta' - \theta^*)^2 - (\theta - \theta^*)^2\right| \\ &\leq B\alpha + \frac{1}{2}|(\theta' - \theta)(-2\theta^* + (\theta' + \theta))| \\ &\leq B\alpha + \frac{1}{2}\alpha(2R) = \alpha(B + R). \end{aligned}$$

Thus, for $\alpha = \frac{\varepsilon}{2R}$, this is at most ε. If we partition $[\theta^* - R, \theta^* + R]$ into length-α intervals, we get a size $O(R^2/\varepsilon) = O(A/\varepsilon)$ partition P_1 of $\mathbb{R}$ that has the desired property for all $y > 0$.

Our final partition is defined by all endpoints in either P_0 and P_1. This has size $O(A/\varepsilon)$, and within each interval the conclusion holds for both $y = 0$ and $y > 0$, as needed.

□

Lemma B.4. *Let $x_1, \dots, x_n$ be fixed, and $y_i \sim \phi(x_i^T w^* + \eta_i)$ for $\eta_i \sim \mathcal{N}(0, 1)$. For $n \geq \frac{1}{\varepsilon^2} k \log \frac{1}{\varepsilon}$, the MLE $\widehat{w}$ satisfies*

$$\widetilde{d}(\widehat{w}, w^*) \leq \varepsilon.$$

Proof. For any $w \in \mathbb{R}^k$, and a sample (x_i, y_i), let $p_w(y|x_i)$ be the conditional distribution of $y = \phi(\langle x_i, w\rangle + \eta)$, and let $\gamma_{i,w}$ be the log-likelihood ratio between w and w^* on this sample:

$$\gamma_{i,w}(y) := \log \frac{p_w(y|x_i)}{p_{w^*}(y|x_i)}.$$

Then

$$\mathop{\mathbb{E}}_y[\gamma_{i,w}(y)] = -d_{KL}(p_{i,w^*}(y|x_i) || p_{i,w}(y|x_i)).$$

Define

$$d_{KL}(w^*, w) := \frac{1}{n} \sum_{i=1}^{n} d_{KL}(p_{i,w^*}(y|x_i) || p_{i,w}(y|x_i)).$$

Concentration. From Lemma B.1, we see that if $\widetilde{d}(w^*, w) \geq \varepsilon$, then for $n \geq O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta})$,

$$\overline{\gamma}_w := \frac{1}{n} \sum_{i=1}^{n} \gamma_{i,w}(y_i) < -\frac{\varepsilon^2}{4}, \tag{18}$$

with probability $1 - \delta$.

Of course, whenever $\overline{\gamma}_w < 0$, the likelihood under w^* is larger than the likelihood under w. Thus, for each *fixed* w with $\widetilde{d}(w^*, w) \geq \varepsilon$, maximizing likelihood would prefer w^* to w with probability $1 - \delta$ if $n \geq O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta})$.

Nothing above is specific to our ReLU-based distribution. But to extend to the MLE over all w, we need to build a net using properties of our distribution.

Building a net. First, with high probability, $|\eta_i| \le B = O(\sqrt{\log n})$ for all i. Suppose this happens. For each i, by an abuse of notation, let $\gamma_{i,w}(y) = \gamma_{\langle x_i, w\rangle}(y)$ where the value of θ^* when considering i is $\langle x_i, w^*\rangle$. By Lemma B.2,

$$\gamma_{i,w}(y_i) \le B^2/2$$

for all i. Let $A = O(n \log n) > nB^2$. By Lemma B.3, for each $i \in [n]$, there exists a partition P_i of $\mathbb{R}$ into $O(A/\varepsilon^2)$ intervals, such that for interval $I \in P_i$, and any w, w' with $x_i^T w, x_i^T w' \in I$, either

$$|\gamma_{i,w}(y_i) - \gamma_{i,w'}(y_i)| \le \varepsilon^2/2 \tag{19}$$

or $\gamma_{i,w}(y_i) < -A$.

These individual partitions P_i on $\langle x_i, w\rangle$ induce a partition P on $\mathbb{R}^k$, where w, w' lie in the same cell of P if $\langle x_i, w\rangle$ and $\langle x_i, w^*\rangle$ are in the same cell of P_i for all $i \in [n]$. Since P is defined by n sets of $O\big(\frac{A}{\varepsilon^2}\big)$ parallel hyperplanes in $\mathbb{R}^k$, the number of cells in P is:

$$2\left(\frac{2Aen}{\varepsilon^2 k}\right)^k.$$

We choose a net $\mathcal{N}$ to contain, for each cell in P, the w in the cell maximizing $\widetilde{d}(w^*, w)$. This has size

$$\log|\mathcal{N}| \lesssim k \log \frac{n}{\varepsilon}.$$

By (18), for our $n \ge O\big(\frac{1}{\varepsilon^2} k \log \frac{k}{\varepsilon}\big)$, we have with high probability that $\overline{\gamma}_w \le -\frac{\varepsilon^2}{4}$, for all $w \in \mathcal{N}$ with $\widetilde{d}(w^*, w) \ge \varepsilon$. Suppose that both this happens, and $|\eta_i| \le B$ for all i. We claim that the MLE $\widehat{w}$ must have $\widetilde{d}(w^*, \widehat{w}) < \varepsilon$.

Consider any $w \in \mathbb{R}^d$ with $\widetilde{d}_{TV}(w^*, w) \ge \varepsilon$. Let $w' \in \mathcal{N}$ lie in the same cell of P. By our choice of $\mathcal{N}$, we know $\widetilde{d}_{TV}(w^*, w') \ge \widetilde{d}_{TV}(w^*, w) \ge \varepsilon$, so $\overline{\gamma}_{w'} \le -\varepsilon^2$. Now we consider two cases. In the first case, there exists i with $\gamma_{i,w}(y_i) < -A$. Then

$$\overline{\gamma}_w = \frac{1}{n} \sum_i \gamma_{i,w}(y_i) \le -\frac{A}{n} + B^2/2 < 0.$$

Otherwise, by (19),

$$\overline{\gamma}_w \le \overline{\gamma}_{w'} + |\overline{\gamma}_w - \overline{\gamma}_{w'}| \le -\varepsilon^2 + \max_i |\gamma_{i,w}(y_i) - \gamma_{i,w'}(y_i)| \le -\varepsilon^2/2.$$

In either case, $\overline{\gamma}_w < 0$ and the likelihood under w^* exceeds that under w. Hence the MLE $\widehat{w}$ must have $\widetilde{d}(w^*, \widehat{w}) \le \varepsilon$. □

Theorem B.5. *Let $y = \phi\big(x^T w^* + \eta\big)$, for $w^* \in \mathbb{R}^k$, $x \sim \mathcal{D}_x$, and $\eta \sim \mathcal{N}(0, 1)$. Then for a sufficiently large constant $C > 0$,*

$$n = C \cdot \frac{k}{\varepsilon^2} \log \frac{1}{\varepsilon}$$

samples of $\{(y_i, x_i)\}_{i=1}^n$ suffices to guarantee that the MLE $\widehat{w}$ satisfies

$$d(\widehat{w}, w^*) \le \varepsilon.$$

Proof. Let D_x denote the dataset $\{x_i\}_{i\in[n]}$ that is used to find the MLE. Notice that the MLE is found using this finite subset, but we would like to make a claim about $\mathcal{D}_x$ without making any parametric or simplifying assumptions on the distribution $\mathcal{D}_x$.

An application of Lemma 4.3 tells us that with probability $1 - e^{-\Omega(n\varepsilon^2)}$, the expectation over the distribution $\mathcal{D}_x$ and the dataset D_x are within $\varepsilon/2$ of one another:

$$d(\widehat{w}, w^*) \le \widetilde{d}(\widehat{w}, w^*) + \varepsilon/2.$$

Now, all we need to show is that the MLE has a small TV distance on the finite dataset, and Lemma B.4 tells us that with probability $1 - e^{-\Omega(n\varepsilon^2)}$,

$$\widetilde{d}(\widehat{w}, w^*) \le \varepsilon/2.$$

Substituting in the above inequality, we get $d(\widehat{w}, w^*) \le \varepsilon$. □

C ReLU Activations with $d > 1$, Unknown Covariance

We recommend the reader review Appendix B, which contains the proof recipe for the case of scalar y. The proofs in this section generalize those of Appendix B.

Consider a sample $(x, y) \in \mathbb{R}^{k \times d}$, with

$$y = \phi(W^* x + \eta), \tag{20}$$

where $W^* \in \mathbb{R}^d \times k$, and noise $\eta \sim \mathcal{N}(0, \Sigma^*)$. The matrices W^* and Σ^* are unknown. For each matrix $W \in \mathbb{R}^{d \times k}$, let $\theta = Wx \in \mathbb{R}^d$, denote a reparametrization of W, and let θ^* denote $\theta^* = W^* x$. Let S denote the co-ordinates of y that are zero-valued. Then the log-likelihood for each θ, Σ is given by

$$f_{\theta,\Sigma}(y) := \log p_{W,\Sigma}(y \mid x) = c - \frac{1}{2}\log|\Sigma| + \log \int_{t: t_S \le 0, t_{S^c} = y_{S^c}} \exp\Big\{-(t-\theta)^T \Sigma^{-1}(t-\theta)/2\Big\}.$$

where c is a normalization constant which does not depend on θ or Σ. Let

$$P := \Sigma^{-1}$$

and let P^* be the precision matrix of the noise η, and $P_S, P_{SS^c}, P_{S^cS}, P_{S^c}$ be the block matrices of P corresponding to the index sets S and its complement S^c.

By some arithmetic involving completion of squares, we can decompose the integral in f into the sum of two functions g, h, such that

$$f_{\theta,\Sigma}(y) = c - \frac{1}{2}\log|\Sigma| + g_{\theta,\Sigma}(y) + h_{\theta,\Sigma}(y).$$

The first term g corresponds to the quadratic term involving the observed positive-valued coordinates y_{S^c}:

$$g_{\theta,\Sigma}(y) = -(y_{S^c} - \theta_{S^c})^T (P_{S^c} - P_{S^cS}(P_S)^{-1}P_{SS^c})(y_{S^c} - \theta_{S^c})/2.$$

As the matrix $P_{S^c} - P_{S^cS}(P_S)^{-1}P_{SS^c} = ((P^{-1})_{S^c})^{-1} = \Sigma_{S^c}^{-1}$ is the precision matrix of η_S, if Σ were the covariance of η, we can simplify the above equation as

$$g_{\theta,\Sigma}(y) = -(y_{S^c} - \theta_{S^c})^T (\Sigma_{S^c})^{-1}(y_{S^c} - \theta_{S^c})/2. \tag{21}$$

The second term corresponds to the probability under θ, P of observing zero-valued coordinates corresponding to the index set S, given the positive coordinates y_{S^c}:

$$h_{\theta,\Sigma}(y) = \log \int_{t \le 0} \exp\Big(-\|P_S^{\frac{1}{2}}(t - \theta_S) + (P_S)^{-1/2}P_{SS^c}(y_{S^c} - \theta_{S^c})\|^2/2\Big). \tag{22}$$

The *log-likelihood ratio* is the difference between $f_{\theta,\Sigma}$ and f_{θ^*,Σ^*}, which we denote by

$$\gamma_{\theta,\Sigma}(y) := f_{\theta,\Sigma}(y) - f_{\theta^*,\Sigma^*}(y)$$

Over a dataset $\{(x_i, y_i)\}_{i \in [n]}$, the average log-likelihood ratio is given by

$$\bar{\gamma}_{W,\Sigma} := \frac{1}{n}\sum_i \gamma_{Wx_i,\Sigma}(y_i).$$

Remark C.1. *For ease of analysis, we will interchange between the precision matrix P in $\gamma_{\theta,P}$ and the covariance matrix Σ in $\gamma_{\theta,\Sigma}$, and it should be understood that $P = \Sigma^{-1}$. The same applies to the functions $g_{\theta,\Sigma}$ and $h_{\theta,\Sigma}$. Finally, the matrix P^* refers to the ground truth precision matrix ($= \Sigma^{*-1}$).*

Analogous to Appendix B, we start by showing that the log-likelihood ratio is bounded by the noise in the sample. The proofs of results in this Section are in Subsection C.1.

Lemma C.2. *Assume $P^* := \Sigma^{*-1}$ satisfies Assumption 4.4.*

For all $y = \phi(\theta^* + \eta)$ such that S denotes the zero-coordinates of y, and η such that $\|P_S^{*\frac{1}{2}}\eta_S\|, \|P_{S^c}^{*\frac{1}{2}}\eta_{S^c}\| \leq B$, if the max eigenvalue $\lambda_{\max}(P)$ satisfies

$$\frac{\lambda_{\max}(P)}{\lambda_{\min}(P^*)} \leq C,$$

then for all $\theta \in \mathbb{R}^d$, we have

$$\gamma_{\theta,P} \leq \frac{d}{2}\log(C) + 3B^2.$$

For the ease of stating the next Lemma, we assume that across the samples of y in the training data, at least one coordinate has sufficiently many positive samples. The proof of our theorem separately handles cases violating this assumption.

Assumption C.3. *Let $\delta \in (0,1)$ be a parameter corresponding to the failure probability of our algorithm. Then, there exists a coordinate $j \in [d]$, such that for at least $n' = O\big(\log\frac{1}{\delta}\big)$ samples $y_{i_1}, \ldots, y_{i_{n'}}$ in the dataset, the j-th coordinate is positive.*

This is a very weak assumption: if it is violated, then $W = 0_{d\times k}, \Sigma = 0$ will achieve a TV distance smaller than $\frac{2\varepsilon^2}{d}$.

Appendix B assumed that the variance in y was 1. Since Section 4.2 considers an unknown Σ^*, we need the following Lemma to show that the MLE will select a precision matrix P, whose eigenvalues are reasonably bounded wrt Σ^{*-1}.

Lemma C.4. *Under Assumption 4.4, C.3, consider $P \in \mathbb{R}_+^{d\times d}$ such that $\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \leq \kappa$ and*

$$\frac{\lambda_{\max}(P)}{\lambda_{\max}(P^*)} \geq O\bigg(\frac{\kappa^3 d^2 n^2}{k^2} + \frac{B^2 n\kappa}{k}\bigg).$$

Then, for all $W \in \mathbb{R}^{d\times k}$, and for all $y_i = \phi(W^* x_i + \eta_i)$ with $\|P_{S^c}^{*\frac{1}{2}}\eta_{S^c}\|, \|P_S^{*\frac{1}{2}}\eta_S\| \leq B$, we have

$$\bar{\gamma}_{W,P} := \frac{1}{n}\sum_{i\in[n]} \gamma_{Wx_i,P}(y_i) < 0.$$

Lemma C.2 and Lemma C.4 show that the MLE will only select precision matrices P that have max eigenvalues in a certain range of the true precision matric P^*.

Now, for matrices in the above eigenvalue range, we first construct a geometric net over the max eigenvalue ρ of the precision matrix, and then cover the matrices whose max eigenvalue is smaller than ρ.

Lemma C.5 (Σ cover)**.** *For $B > 1$, and $0 < L < U$, let $A > \max\Big\{\sqrt{\log\frac{1}{\varepsilon}}, B^2 U\kappa, \frac{d}{2}\log\big(\frac{\kappa U}{L}\big), 1\Big\}$.*

Let $P^* := \Sigma^{*-1}$ be the precision matrix of η. Let $\Omega \subset \mathbb{R}_+^{d\times d}$ denote the set of positive definite matrices $P \in \mathbb{R}_+^{d\times d}$ with condition number κ and whose maximum eigenvalue lies in $[L\lambda_{\min}(P^*), U\cdot\lambda_{\max}(P^*)]$.

Then, there exists a partition of Ω of size

$$\left(\mathrm{poly}\left(A, \frac{1}{\varepsilon}\right)\right)^{d^2}$$

such that for all $\theta \in \mathbb{R}^d$ and all $y = \phi(\theta^* + \eta) \in \mathbb{R}^d$ with $\|P_S^{*\frac{1}{2}}\eta_S\|, \|P_{S^c}^{*\frac{1}{2}}\eta_{S^c}\| \leq B$, and each cell I in the partition, one of the following holds:

- *for all $P \in I$, $\gamma_{\theta,P}(y) < -A$, or*
- *for all $P, P' \in I$, we have $|\gamma_{\theta,P}(y) - \gamma_{\theta,P'}(y)| \leq \epsilon$.*

Analogous to Appendix B, we now construct a partition over W for a fixed precision matrix P, such that each cell in the partition has very small log-likelihood (in which case the MLE will not choose it) or the log-likelihood changes slowly.

Lemma C.6 (W-net). *Let η_{S^c}, η_S be such that*

$$\|P_{S^c}^{*\frac{1}{2}} \eta_{S^c}\| \leq B_1, \|P_S^{*\frac{1}{2}} \eta_S\| \leq B_2,$$

for $B_1, B_2 \geq 0$.

Let $A > \max\{B_1^2, B_2^2, \text{poly}(C, \kappa)\}$. Let $P^* = \Sigma^{*-1}$ be the precision matrix of η. For a fixed matrix $P \in \mathbb{R}^{d \times d}$ whose condition number satisfies Assumption 4.4 and whose eigenvalues satisfy $\lambda_{\max}(P) \in [e^{-\frac{2A}{d}} \lambda_{\min}(P^*), C\lambda_{\max}(P^*)]$, there exists a partition $\mathcal{I}$ of $\mathbb{R}^d$ with size

$$\left(\text{poly}\left(A, \frac{1}{\varepsilon}\right)\right)^{3d}$$

such that for each interval $I \in \mathcal{I}$, we have one of the following:

- *for all $\theta \in I$, $\gamma_{\theta,P}(y) < -A$, or*
- *for all $\theta, \theta' \in I$, $|\gamma_{\theta,P}(y) - \gamma_{\theta',P}(y)| \leq \epsilon$.*

Using the above lemmas, we can show that the MLE will only pick out $\widehat{W}, \widehat{P}$ such that they have small TV on the dataset of $\{x_i\}$.

Lemma C.7. *Let $x_1, \dots, x_n$ be fixed, and $y_i = \phi(W^* x_i + \eta_i)$ for $\eta_i \sim \mathcal{N}(0, \Sigma^*)$, and $W^* \in \mathbb{R}^{d \times k}$ with $\Sigma^* \in \mathbb{R}^{d \times d}$ satisfying Assumption 4.4 and Assumption C.3. For a sufficiently large constant $C > 0$,*

$$n = C \cdot \frac{(d^2 + kd)}{\varepsilon^2} \log \frac{kd\kappa}{\varepsilon}$$

samples suffice to guarantee that with high probability, the MLE $\widehat{W}, \widehat{\Sigma}$ satisfies

$$\widetilde{d}\Big((\widehat{W}, \widehat{\Sigma}), (W^*, \Sigma^*)\Big) \leq \varepsilon.$$

Lemma C.8. *Let $\{x_i\}_{i=1}^n$ be i.i.d. random variables such that $x_i \sim \mathcal{D}_x$.*

Let $P^* := \Sigma^{*-1}$. Let $\lambda_{\min}^*, \lambda_{\max}^*$ be the minimum and maximum eigenvalues of P^*. For $0 < L < U$, let Ω denote the following set of precision matrices

$$\Omega := \left\{ P \in \mathbb{R}_+^{d \times d} : \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \leq \kappa \text{ and } \lambda_{\max}(P) \in [L \cdot \lambda_{\min}^*, U \cdot \lambda_{\max}^*] \right\}.$$

Then, for a sufficiently large constant $C > 0$, and for

$$n = C \cdot \left(\frac{kd + d^2}{\varepsilon^2}\right) \log\left(\frac{kd\kappa}{\varepsilon} \log\left(\frac{U}{L}\right)\right),$$

we have:

$$\Pr_{x_i \sim \mathcal{D}_x}\left[\sup_{W \in \mathbb{R}^{d \times k}, P \in \Omega} \Big| \widetilde{d}((W, P), (W^*, P^*)) - d((W, P), (W^*, P^*)) \Big| > \varepsilon\right] \leq e^{-\Omega(n\varepsilon^2)}.$$

Theorem 4.5. *Let $\mathbb{R}_\kappa^{d \times d}$ denote the set of positive definite matrices with condition number κ. Given n samples $\{(x_i, y_i)\}_{i=1}^n$ satisfying Assumption 4.4, where $x_i \sim \mathcal{D}_x$ i.i.d., and y_i is generated according to (7), let $\widehat{W}, \widehat{\Sigma} := \arg\max_{W \in \mathbb{R}^{d \times k}, \Sigma \in \mathbb{R}_\kappa^{d \times d}} \frac{1}{n} \sum_i \log p_{W,\Sigma}(y_i \mid x_i)$. Then, for a sufficiently large constant $C > 0$,*

$$n = C \cdot \left(\frac{kd + d^2}{\varepsilon^2}\right) \log\left(\frac{\kappa kd}{\varepsilon\delta}\right)$$

samples suffice to ensure that with probability $1 - \delta$, we have

$$d_{TV}\Big((\widehat{W}, \widehat{\Sigma}), (W^*, \Sigma^*)\Big) \leq \varepsilon.$$

Proof of Theorem 4.5. First, we consider the cases violating Assumption C.3.

As $n \propto \frac{d^2}{\varepsilon^2} \log \frac{1}{\delta}$, if assumption C.3 is violated, then it implies that each coordinate is non-zero in atmost a ε^2/d^2 fraction of the samples, and a union bound implies that the probability of seeing a non-zero vector is atmost ε^2/d. Hence, with high probability over the draws of the data, returning the all-zeros vector always will achieve a TV distance smaller than $\frac{2\varepsilon^2}{d}$.

Let $\widehat{P}, P^* = \widehat{\Sigma}^{-1}, \Sigma^{*-1}$. Now, if Assumption C.3 holds, Lemma C.7 guarantees that the MLE has small TV on the x_i observed in the dataset:

$$\widetilde{d}((\widehat{W}, \widehat{P}), (W^*, P^*)) \leq \varepsilon.$$

The above result is over the finite x_i observed in our dataset. To generalize it over $x \sim \mathcal{D}_x$, we use Lemma C.8, which gives

$$d((\widehat{W}, \widehat{P}), (W^*, P^*)) - \widetilde{d}((\widehat{W}, \widehat{P}), (W^*, P^*)) \leq \varepsilon.$$

Rescaling ε gives the conclusion of the Theorem. □

C.1 Proofs of Appendix C.

Lemma C.2. *Assume $P^* := \Sigma^{*-1}$ satisfies Assumption 4.4.*

For all $y = \phi(\theta^* + \eta)$ such that S denotes the zero-coordinates of y, and η such that $\|P_S^{*\frac{1}{2}} \eta_S\|, \|P_{S^c}^{*\frac{1}{2}} \eta_{S^c}\| \leq B$, if the max eigenvalue $\lambda_{\max}(P)$ satisfies

$$\frac{\lambda_{\max}(P)}{\lambda_{\min}(P^*)} \leq C,$$

then for all $\theta \in \mathbb{R}^d$, we have

$$\gamma_{\theta,P} \leq \frac{d}{2} \log(C) + 3B^2.$$

Proof. We have

$$\gamma_{\theta,\Sigma} \leq \frac{1}{2} \log \frac{|\Sigma^*|}{|\Sigma|} + g_{\theta,\Sigma} - g_{\theta^*,\Sigma^*} + h_{\theta,\Sigma} - h_{\theta^*,\Sigma^*}. \tag{23}$$

From Lemma C.9, C.10, we have

$$g_{\theta,\Sigma} - g_{\theta^*,\Sigma^*} + h_{\theta,\Sigma} - h_{\theta^*,\Sigma^*} \leq g_{\theta,\Sigma} + \frac{1}{2} \log \frac{|P_S^*|}{|P_S|} + 3B^2.$$

Substituting in Eqn (23), we get

$$\gamma_{\theta,\Sigma} \leq g_{\theta,\Sigma} + \frac{1}{2} \log \frac{|\Sigma^*|}{|\Sigma|} + \frac{1}{2} \log \frac{|P_S^*|}{|P_S|} + 3B^2.$$

As $(P_S^*)^{-1} = \Sigma_S^* - \Sigma_{SS^c}^* \Sigma_{S^c}^{*-1} \Sigma_{S^cS}^*$, by the matrix determinant rule, we have

$$\log|\Sigma^*| + \log|P_S^*| = \log|\Sigma_{S^c}^*|.$$

This gives

$$\gamma_{\theta,\Sigma} \leq g_{\theta,\Sigma} + \frac{1}{2} \log \frac{|\Sigma_{S^c}^*|}{|\Sigma_{S^c}|} + 3B^2.$$

This gives

$$\begin{aligned} \gamma_{\theta,\Sigma} &\leq g_{\theta,\Sigma} + \frac{d}{2} \log \frac{\lambda_{\max}(\Sigma^*)}{\lambda_{\min}(\Sigma)} + 3B^2, \\ &= g_{\theta,\Sigma} + \frac{d}{2} \log \frac{\lambda_{\max}(P)}{\lambda_{\min}(P^*)} + 3B^2. \end{aligned} \tag{24}$$

As the matrix $\Sigma_{S^c}^{-1}$ is positive definite, we trivially get

$$g_{\theta_i,\Sigma}(y) = -(y_{i,S^c} - \theta_{i,S^c})^T (\Sigma_{S^c})^{-1} (y_{i,S^c} - \theta_{i,S^c})/2 \leq 0.$$

Substituting in Eqn (24), we get

$$\gamma_{\theta,\Sigma} \leq \frac{d}{2} \log \frac{\lambda_{\max}(P)}{\lambda_{\min}(P^*)} + 3B^2.$$

As the Lemma assumes

$$\lambda_{\max}(P) \leq C\lambda_{\min}(P^*),$$

we get

$$\gamma_{\theta,\Sigma} \leq \frac{d}{2} \log(C) + 3B^2.$$

□

Lemma C.9. *Consider the function g defined in Eq (21). For the ground truth parameters θ^*, Σ^*, the function g_{θ^*,Σ^*} satisfies*

$$-g_{\theta^*,\Sigma^*} \leq \frac{1}{2} \|\eta_{S^c}\|^2_{\Sigma_{S^c}},$$

which is, with probability $1 - e^{-\Omega(d)}$,

$$-g_{\theta^*,\Sigma^*} \leq O(d).$$

Proof. As y_{S^c} are the positive valued coordinates in y, we have

$$y_{S^c} - \theta^*_{S^c} = \eta_{S^c},$$

which gives

$$\begin{aligned} g_{\theta^*,\Sigma^*}(y) &= -(y_{S^c} - \theta^*_{S^c})^T (\Sigma^*_{S^c})^{-1} (y_{S^c} - \theta^*_{S^c})/2, \\ &= -\|\eta_{S^c}\|^2_{\Sigma_{S^c}}/2. \end{aligned}$$

As η_{S^c} is Gaussian with covariance $\Sigma^*_{S^c}$, the expected norm is $\frac{|S^c|}{2}$, which implies that with probability $1 - e^{-\Omega(|S^c|)}$, we have

$$-g_{\theta^*,\Sigma^*}(y) \leq O(|S^c|).$$

□

Lemma C.10. *Consider y generated according to Eqn (20) by*

$$y = \phi(\theta^* + \eta), \quad \eta \sim \mathcal{N}(0, \Sigma^*).$$

For all $\theta \in \mathbb{R}^d, \Sigma \in \mathbb{R}^{d\times d}_+$, and the function $h_{\theta,\Sigma}$ defined in Eqn (22), the difference $h_{\theta,\Sigma}(y) - h_{\theta^*,\Sigma^*}(y)$ satisfies

$$h_{\theta,\Sigma}(y) - h_{\theta^*,\Sigma^*}(y) \leq \frac{1}{2} \log \frac{|P^*_S|}{|P_S|} + \|P^{*\frac{1}{2}}_{S^c} \eta_{S^c}\|^2 + 2\|P^{*\frac{1}{2}}_S \eta_S\|^2 - \|\eta_{S^c}\|^2_{\Sigma_{S^c}} + O(|S|), \tag{25}$$

where $P^* = \Sigma^{*-1}$ is the precision matrix of η.

Proof. For $\theta \in \mathbb{R}^d, \Sigma \in \mathbb{R}^{d\times d}_+$, and $P = \Sigma^{-1}$, we have

$$\begin{aligned} h_{\theta,\Sigma}(y) &= \log \int_{t\leq 0} \exp\Big(-\|P_S^{\frac{1}{2}}(t - \theta_S) + (P_S)^{-1/2} P_{SS^c}(y_{S^c} - \theta_{S^c})\|^2/2\Big), \\ &\leq \log \int_{t\in\mathbb{R}^{|S|}} \exp\Big(-\|P_S^{\frac{1}{2}}(t - \theta_S) + (P_S)^{-1/2} P_{SS^c}(y_{S^c} - \theta_{S^c})\|^2/2\Big), \\ &\leq \frac{|S|}{2} \log(2\pi) - \frac{1}{2} \log|P_S|, \end{aligned} \tag{26}$$

where the last step follows from the integral of a Gaussian pdf. This gives a sufficient upper bound on $h_{\theta,\Sigma}(y)$, and now we will focus on lower bounding $h_{\theta^*,\Sigma^*}(y)$.

For the coordinates of y in S^c, we have $y_{S^c} - \theta^*_{S^c} = \eta_{S^c}$. Substituting in Eqn (22), we get

$$h_{\theta^*,\Sigma^*}(y) = \log \int_{t\le 0} \exp\Big(-\|P_S^{*\frac{1}{2}}(t-\theta^*_S) + (P^*_S)^{-1/2} P^*_{SS^c}\eta_{S^c}\|^2/2\Big),$$
$$= \log \int_{t\le 0} \exp\Big(-\|P_S^{*\frac{1}{2}}(t-\theta^*_S) + (P^*_S)^{-1/2} P^*_{SS^c}\eta_{S^c}\|^2/2\Big).$$

Using $\|a+b\|^2 \le 2a^2 + 2b^2$, we get

$$h_{\theta^*,\Sigma^*}(y) \ge -\|(P^*_S)^{-1/2} P^*_{SS^c}\eta_{S^c}\|^2 + \log \int_{t\le 0} \exp\Big(-\|P_S^{*\frac{1}{2}}(t-\theta^*_S)\|^2\Big).$$

Set $u := P_S^{*\frac{1}{2}}(t-\theta^*_S)$, and by the change of variables formula, we get:

$$h_{\theta^*,\Sigma^*}(y) \ge -\|(P^*_S)^{-1/2} P^*_{SS^c}\eta_{S^c}\|^2 + \log \int_{P_S^{*-\frac{1}{2}}u+\theta^*_S \le 0} \Big|(P^*_S)^{-1/2}\Big| \cdot \exp(-\|u\|^2),$$
$$= -\|(P^*_S)^{-1/2} P^*_{SS^c}\eta_{S^c}\|^2 + \frac{1}{2}\log\big|P_S^{*-1}\big| + \log \int_{P_S^{*-\frac{1}{2}}u+\theta^*_S \le 0} \exp(-\|u\|^2).$$

For $i \in S$, we have $\theta^*_i + \eta_i \le 0$. This gives

$$P_S^{*-\frac{1}{2}}u \le \eta_S \Rightarrow P_S^{*-\frac{1}{2}}u + \theta^*_S \le 0 \Rightarrow \log \int_{P_S^{*-\frac{1}{2}}u+\theta^*_S \le 0} \exp(-\|u\|^2) \ge \log \int_{P_S^{*-\frac{1}{2}}u \le \eta_S} \exp(-\|u\|^2),$$

using which we get

$$h_{\theta^*,\Sigma^*}(y) \ge -\|(P_S)^{*-1/2} P^*_{SS^c}\eta_{S^c}\|^2 - \frac{1}{2}\log|P^*_S| + \log \int_{P_S^{*-\frac{1}{2}}u \le \eta_S} \exp(-\|u\|^2).$$

By another change of variables via $v := P_S^{*-\frac{1}{2}}u - \eta_S$, we get

$$h_{\theta^*,\Sigma^*}(y) \ge -\|(P^*_S)^{-1/2} P^*_{SS^c}\eta_{S^c}\|^2 - \frac{1}{2}\log|P^*_S| + \log \int_{v\le 0} \Big|P_S^{*\frac{1}{2}}\Big| \exp\Big(-\|P_S^{*\frac{1}{2}}(v+\eta_S)\|^2\Big),$$
$$\ge -\|(P^*_S)^{-1/2} P^*_{SS^c}\eta_{S^c}\|^2 - \frac{1}{2}\log|P^*_S| - 2\|P_S^{*\frac{1}{2}}\eta_S\|^2 + \log \int_{v\le 0} \Big|P_S^{*\frac{1}{2}}\Big| \exp\Big(-2\|P_S^{*\frac{1}{2}}v\|^2\Big),$$
$$= -\|(P^*_S)^{-1/2} P^*_{SS^c}\eta_{S^c}\|^2 - \frac{1}{2}\log|P^*_S| - 2\|P_S^{*\frac{1}{2}}\eta_S\|^2 + O(|S|).$$

As $(\Sigma^*_{S^c})^{-1} = P^*_{S^c} - P^*_{S^cS}(P^*_S)^{-1}P^*_{SS^c}$, we have

$$-\|(P_S)^{*-1/2} P^*_{SS^c}\eta_{S^c}\|^2 = \|\eta_{S^c}\|^2_{\Sigma^*_{S^c}} - \|P_{S^c}^{*1/2}\eta_{S^c}\|^2,$$

which gives

$$h_{\theta^*,\Sigma^*}(y) \ge \|\eta_{S^c}\|^2_{\Sigma^*_{S^c}} - \|P_{S^c}^{*\frac{1}{2}}\eta_{S^c}\|^2 - \frac{1}{2}\log|P^*_S| - 2\|P_S^{*\frac{1}{2}}\eta_S\|^2 + O(|S|). \tag{27}$$

From Eqn (26) $-$ Eqn (27), we get

$$h_{\theta,\Sigma}(y) - h_{\theta^*,\Sigma^*}(y) \le \|P_{S^c}^{*\frac{1}{2}}\eta_{S^c}\|^2 + 2\|P_S^{*\frac{1}{2}}\eta_S\|^2 - \|\eta_{S^c}\|^2_{\Sigma^*_{S^c}} + \frac{1}{2}\log\frac{|P^*_S|}{|P_S|} + O(|S|). \tag{28}$$

□

Lemma C.4. *Under Assumption 4.4, C.3, consider $P \in \mathbb{R}_+^{d\times d}$ such that $\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \le \kappa$ and*

$$\frac{\lambda_{\max}(P)}{\lambda_{\max}(P^*)} \ge O\bigg(\frac{\kappa^3 d^2 n^2}{k^2} + \frac{B^2 n\kappa}{k}\bigg).$$

Then, for all $W \in \mathbb{R}^{d\times k}$, and for all $y_i = \phi(W^* x_i + \eta_i)$ with $\|P_{S^c}^{*\frac{1}{2}}\eta_{S^c}\|, \|P_S^{*\frac{1}{2}}\eta_S\| \le B$, we have

$$\bar{\gamma}_{W,P} := \frac{1}{n}\sum_{i\in[n]} \gamma_{Wx_i,P}(y_i) < 0.$$

Proof of Lemma C.4. For each $W \in \mathbb{R}^{d\times k}$, let

$$\theta_i := Wx_i.$$

From Eqn (24) in Lemma C.2, for each $i \in [n]$, we have,

$$\begin{aligned}\gamma_{\theta_i,\Sigma} &\le g_{\theta_i,\Sigma} + \frac{d}{2}\log\frac{\lambda_{\max}(P)}{\lambda_{\min}(P^*)} + 3B^2,\\ &\le g_{\theta_i,\Sigma} + \frac{d}{2}\log\frac{\kappa\lambda_{\max}(P)}{\lambda_{\max}(P^*)} + 3B^2.\end{aligned}$$

Now consider

$$\begin{aligned}g_{\theta_i,\Sigma}(y) &= -\frac{1}{2}(y_{i,S^c} - \theta_{i,S^c})^T(\Sigma_{S^c})^{-1}(y_{i,S^c} - \theta_{i,S^c}),\\ &\le -\frac{1}{2}\|y-\theta\|^2\lambda_{\min}(\Sigma_{S^c}^{-1}),\\ &\le -\frac{1}{2}\|y-\theta\|^2\lambda_{\min}(\Sigma^{-1}) = -\frac{1}{2}\|y-\theta\|^2\lambda_{\min}(P),\\ &\le -\frac{1}{2}\|y-\theta\|^2\frac{\lambda_{\max}(P)}{\kappa},\end{aligned}$$

where the second inequality comes from the eigenvalue interlacing Theorem, and the last line follows from the condition number assumption on Σ, P.

By Assumption C.3, there exist at least $\varepsilon^2 n$ samples for a coordinate j such that $(y_i)_j > 0$. Averaging $g_{\theta_i,\Sigma}$, by Lemma A.1, we get that with high probability,

$$\sum_i \|y_i - \theta_i\|^2 \ge \frac{\sigma_j^{*2}k}{2},$$

which gives

$$\begin{aligned}\frac{1}{n}\sum_i g_{\theta_i,\Sigma}(y_i) &\le -\frac{\sigma_j^{*2}k\lambda_{\max}(P)}{4n\kappa},\\ &\le -\frac{k\lambda_{\max}(P)}{4n\kappa\lambda_{\max}(P^*)}.\end{aligned}$$

This gives

$$\begin{aligned}\bar{\gamma}_{W,\Sigma} &\le -\frac{\lambda_{\max}(P)k}{4n\kappa\lambda_{\max}(P^*)} + \frac{d}{2}\log\left(\kappa\cdot\frac{\lambda_{\max}(P)}{\lambda_{\max}(P^*)}\right) + 3B^2,\\ &\le -\frac{\lambda_{\max}(P)k}{4n\kappa\lambda_{\max}(P^*)} + d\sqrt{\kappa\cdot\frac{\lambda_{\max}(P)}{\lambda_{\max}(P^*)}} + 3B^2.\end{aligned}$$

Completing the squares, we get

$$\bar{\gamma}_{W,\Sigma} \le -\left(\sqrt{\frac{\lambda_{\max}(P)k}{4n\kappa\lambda_{\max}(P^*)}} - \kappa d\sqrt{\frac{n}{k}}\right)^2 + \frac{\kappa^2 d^2 n}{k} + 3B^2.$$

For

$$\frac{\lambda_{\max}(P)}{\lambda_{\max}(P^*)} \ge O\left(\frac{\kappa^3 d^2 n^2}{k^2} + \frac{B^2 n\kappa}{k}\right),$$

the above inequality satisfies

$$\bar{\gamma}_{W,\Sigma} \le 0.$$

□

Lemma C.11. *Assume $P^* := \Sigma^{*-1}$ satisfies Assumption 4.4 with condition number κ.*

For all $y = \phi(\theta^* + \eta)$ such that S denotes the zero-coordinates of y, and η such that $\|P_S^{*\frac{1}{2}}\eta_S\|, \|P_{S^c}^{*\frac{1}{2}}\eta_{S^c}\| \leq B$, consider precision matrices P whose max eigenvalue $\lambda_{\max}(P)$ satisfies

$$\frac{\lambda_{\max}(P)}{\lambda_{\min}(P^*)} \leq C.$$

Let $A \geq 4\max\{\frac{d}{2}\log C, 3B^2\}$. Then for $V := (P^{-1})_{S^c}$ and R_P defined as

$$R_P := 2B\sqrt{C} + \sqrt{\frac{3}{2}A}, \tag{29}$$

we have

$$\|\theta_{S^c} - \theta^*_{S^c}\|_V \geq R_P \implies \gamma_{\theta,P} \leq -A.$$

Proof of Lemma C.11. Consider Eqn (24) in Lemma C.2. We have

$$\begin{aligned}\gamma_{\theta,P} &\leq g_{\theta,P} + \frac{d}{2}\log\frac{\lambda_{\max}(P)}{\lambda_{\min}(P^*)} + 3B^2,\\ &\leq g_{\theta,P} + \frac{d}{2}\log C + 3B^2,\end{aligned}$$

where the last inequality follows from $\frac{\lambda_{\max}(P)}{\lambda_{\min}(P^*)} \leq C$ in the statement of the Lemma.

By the definition of $g_{\theta,P}$, we have

$$g_{\theta,P} := -\frac{1}{2}(y_{S^c} - \theta_{S^c})^T(P_{S^c} - P_{S^cS}P_S^{-1}P_{SS^c})(y_{S^c} - \theta_{S^c}).$$

We can rewrite the matrix $(P_{S^c} - P_{S^cS}P_S^{-1}P_{SS^c})$ as

$$(P_{S^c} - P_{S^cS}P_S^{-1}P_{SS^c}) = \big((P^{-1})_{S^c}\big)^{-1}.$$

By setting

$$V := (P^{-1})_{S^c},$$

we can rewrite $g_{\theta,P}$ as

$$g_{\theta,P} := -\frac{1}{2}\|y_{S^c} - \theta_{S^c}\|_V^2 = -\frac{1}{2}(y_{S^c} - \theta_{S^c})^T V^{-1}(y_{S^c} - \theta_{S^c}).$$

Now, as $y_{S^c} = \eta_{S^c} + \theta^*_{S^c}$, we have

$$\begin{aligned}g_{\theta,P} &= -\frac{1}{2}\|\eta_{S^c} + \theta^*_{S^c} - \theta_{S^c}\|_V^2,\\ &= -\frac{1}{2}\|\theta^*_{S^c} - \theta_{S^c}\|_V^2 + \|\eta_{S^c}\|_V\|\theta^*_{S^c} - \theta_S\| - \frac{1}{2}\|\eta_{S^c}\|_V^2.\end{aligned}$$

Ignoring the $\|\eta_{S^c}\|_V^2$ term, we get

$$g_{\theta,P} \leq -\frac{1}{2}\|\theta^*_{S^c} - \theta_{S^c}\|_V^2 + \|\eta_{S^c}\|_V\|\theta^*_{S^c} - \theta_S\|_V.$$

By the Cauchy-Schwartz inequality and the eigenvalue interlacing theorem, we have

$$\|\eta_{S^c}\|_V \leq \lambda_{\max}^{\frac{1}{2}}(V^{-1}) \cdot \|\eta_{S^c}\|_2 = \frac{\|\eta_{S^c}\|_2}{\lambda_{\min}^{\frac{1}{2}}(V)} = \frac{\|\eta_{S^c}\|_2}{\lambda_{\min}^{\frac{1}{2}}(P_{S^c}^{-1})} \leq \frac{\|\eta_{S^c}\|_2}{\lambda_{\min}^{\frac{1}{2}}(P^{-1})} = \lambda_{\max}^{\frac{1}{2}}(P) \cdot \|\eta_{S^c}\|_2$$

By the statement of the Lemma, we have $\|P^{*\frac{1}{2}}_{S^c}\eta_{S^c}\| \le B \implies \|\eta_{S^c}\|_2 \le \frac{B}{\lambda^{\frac{1}{2}}_{\min}(P^*_{S^c})}$. Substituting in the above inequality, we get

$$\|\eta_{S^c}\|_V \le \frac{\lambda^{\frac{1}{2}}_{\max}(P) \cdot \|\eta_{S^c}\|_2}{\lambda^{\frac{1}{2}}_{\min}(P^*_{S^c})} \le \sqrt{C}B.$$

Substituting in the function $g_{\theta,P}$, we get

$$g_{\theta,P} \le -\frac{1}{2}\|\theta^*_{S^c} - \theta_{S^c}\|_V^2 + B\sqrt{C}\|\theta^*_{S^c} - \theta_S\|_V.$$

Hence, for θ satisfying

$$\|\theta_{S^c} - \theta^*_{S^c}\|_V \ge R_P := 2B\sqrt{C} + 2\sqrt{A},$$

we get

$$\gamma_{\theta,P} \le -A.$$

□

In order to cover our precision matrices, we will consider a subset of matrices whose entries are quantized by an interval size β:

Definition C.12 (Quantized Precision Matrices)**.** *For $\kappa > 0$, define $\Omega \subset \mathbb{R}^{d\times d}$ as the set of positive definite matrices with condition number κ.*

For $\rho > 0$, define the set $\Omega_\rho \subset \Omega$ as

$$\Omega_\rho := \left\{P \in \Omega : \lambda_{\max}(P) \in \left[\frac{\rho}{2}, \rho\right]\right\}$$

For a quantization size $\beta > 0$, define $\widetilde{\Omega}_{\rho,\beta} \subset \Omega_\rho$ as:

$$\widetilde{\Omega}_{\rho,\beta} := \{P \in \Omega_\rho : P_{ij} \in \{-\rho, -\rho(1-\beta), -\rho(1-2\beta), \cdots, \rho(1-2\beta), \rho(1-\beta), \rho\}.\}$$

Lemma C.5 (Σ cover)**.** *For $B > 1$, and $0 < L < U$, let $A > \max\left\{\sqrt{\log\frac{1}{\varepsilon}}, B^2U\kappa, \frac{d}{2}\log(\frac{\kappa U}{L}), 1\right\}$.*

Let $P^* := \Sigma^{*-1}$ be the precision matrix of η. Let $\Omega \subset \mathbb{R}_+^{d\times d}$ denote the set of positive definite matrices $P \in \mathbb{R}_+^{d\times d}$ with condition number κ and whose maximum eigenvalue lies in $[L\lambda_{\min}(P^*), U \cdot \lambda_{\max}(P^*)]$.

Then, there exists a partition of Ω of size

$$\left(\text{poly}\left(A, \frac{1}{\varepsilon}\right)\right)^{d^2}$$

such that for all $\theta \in \mathbb{R}^d$ and all $y = \phi(\theta^* + \eta) \in \mathbb{R}^d$ with $\|P^{*\frac{1}{2}}_S\eta_S\|, \|P^{*\frac{1}{2}}_{S^c}\eta_{S^c}\| \le B$, and each cell I in the partition, one of the following holds:

- *for all $P \in I$, $\gamma_{\theta,P}(y) < -A$, or*
- *for all $P, P' \in I$, we have $|\gamma_{\theta,P}(y) - \gamma_{\theta,P'}(y)| \le \epsilon$.*

Proof. In order to construct the net over the precision matrices, we will consider geometrically spaced values of $\rho \in [L \cdot \lambda_{\min}(P^*), U \cdot \lambda_{\max}(P^*)]$, and for each ρ, we will construct a net over matrices that have max eigenvalue $\le \rho$.

Now consider $\rho > 0$ that lies in the following discrete set:

$$\left\{\lambda_{\min}(P^*)2^j, j \in \lceil\log_2(\kappa\tfrac{U}{L})\rceil\right\}$$

This set is a geometric partition over the possible max eigenvalues that the MLE can return.

For the current ρ, let Ω_ρ follow Definition C.12. Now consider $P \in \Omega_\rho$.

Constructing the interval for which $\gamma_{\theta,P} < -A$. By Lemma C.11, for $V = (P^{-1})_{S^c}$, and $R_P = O\Big(B\sqrt{\frac{\rho}{\lambda_{\min}(P^*)}} + \sqrt{A}\Big) = O(\sqrt{A})$, we have

$$\|\theta_{S^c} - \theta^*_{S^c}\|_V \geq R_P \implies \gamma_{\theta,P} < -A.$$

For any θ, notice that the set of matrices P satisfying $\|\theta_{S^c} - \theta^*_{S^c}\|_V \geq R_P$ is connected (as its complement is compact). This forms the set I for which $\gamma_{\theta,P} < -A$.

Constructing intervals for which $|\gamma_{\theta,P} - \gamma_{\theta,P'}| \leq \varepsilon$. We will now construct a partition over those P which satisfy $\|\theta_{S^c} - \theta^*_{S^c}\|_V < R_P$, and show that the log-likelihood changes by atmost ε for each cell in this partition.

If $P \in \Omega_\rho$, then each of its elements $P_{ij} \in [-\rho, \rho]$. For a parameter $\beta > 0$ that we will specify later, consider the partition $\widetilde{\Omega}_{\rho,\beta}$ of Ω_ρ, following Definition C.12. Clearly, the size of $\widetilde{\Omega}_{\rho,\beta}$ can be upper bounded by

$$\left|\widetilde{\Omega}_{\rho,\beta}\right| \leq \left(\frac{2}{\beta}\right)^{d^2}.$$

We will now analyze the effect of rounding down $P \in \Omega_\rho$ to its nearest element in $\widetilde{\Omega}_{\rho,\beta}$.

By Claim C.13, for $\gamma = 2\kappa\beta d^2$, we have

$$(1-\gamma)\|t-\theta\|^2_\Sigma \leq \|t-\theta\|^2_{\Sigma'} \leq (1+\gamma)\|t-\theta\|^2_\Sigma, \tag{30}$$

Consider the log-likelihood at θ, P':

$$f_{\theta,P'}(y) = \frac{1}{2}\log|P'| + \log\int_{t:t_S\leq 0, t_{S^c}=y_{S^c}} \exp\left(-\|t-\theta\|^2_{\Sigma'}\right).$$

We will use the LHS of Eqn (30) to show that

$$f_{\theta,P'}(y) - \frac{1}{2}\log|P'| \leq f_{\theta,P}(y) - \frac{1}{2}\log|P| + \varepsilon,$$

and deal with the $\log|P'|$ term later. The lower bound for the log-likelihood at P' can be obtained via analogous proof using the RHS of Eqn (30).

By the LHS of Eqn (30), we get

$$f_{\theta,P'}(y) - \frac{1}{2}\log|P'| \leq \log\int_{t:t_S\leq 0, t_{S^c}=y_{S^c}} \exp\left(-(1-\gamma)\|t-\theta\|^2_\Sigma\right).$$

Rearranging the terms, we get

$$\begin{aligned} f_{\theta,P'}(y) - \frac{1}{2}\log|P'| \leq &-\frac{(1-\gamma)}{2}\|y_{S^c} - \theta_{S^c}\|^2_{\Sigma_{S^c}} \\ &+ \log\int_{t\leq 0} \exp\left(-\frac{(1-\gamma)}{2}\|P_S^{\frac{1}{2}}(t-\theta)_S + P_S^{-\frac{1}{2}}P_{SS^c}(y_{S^c}-\theta_{S^c})\|^2\right) \end{aligned}$$

The non-integral term corresponds to $g_{\theta,P}$ in Eqn (21), while the integral term corresponds to $h_{\theta,P}$ in Eqn (22).

Handling the non-integral term. As we are only considering θ such that $\|y_{S^c} - \theta_{S^c}\|_{\Sigma_{S^c}} \leq R_P$, we have that for

$$\beta = O\left(\frac{\varepsilon}{R_P^2 d^2 \kappa}\right) = O\left(\frac{\varepsilon}{\mathrm{poly}(A)}\right),$$

the non-integral term corresponds to $g_{\theta,P} + \varepsilon$, which gives

$$f_{\theta,P'}(y) - \frac{1}{2}\log|P'| \leq g_{\theta,P} + \varepsilon + \log\int_{t\leq 0}\exp\left(-\frac{(1-\gamma)}{2}\|P_S^{\frac{1}{2}}(t-\theta)_S + P_S^{-\frac{1}{2}}P_{SS^c}(y_{S^c}-\theta_{S^c})\|^2\right) \tag{31}$$

Handling the integral. Now we consider the integral term. Define the integral

$$I_1 = \log \int_{t \le 0} \exp\left(-\frac{(1-\gamma)}{2} \|P_S^{\frac{1}{2}}(t-\mu)\|^2\right)$$

for $\gamma = 2d^2\kappa\beta$ and $\mu = \theta_S - P_S^{-1} P_{SS^c}(y_{S^c} - \theta_{S^c})$.

Define the analogous integral that does not have the $(1-\gamma)$ term in the exponential:

$$I_2 = \log \int_{t \le 0} \exp\left(-\frac{1}{2} \|P_S^{\frac{1}{2}}(t-\mu)\|^2\right)$$

Clearly, $I_1 \ge I_2$.

We only need to consider μ such that $\|\mu\|_\infty \le O(\sqrt{A}\rho)$: otherwise the likelihood will be smaller than $-A$.

By Lemma C.14, for $\gamma = O\left(\frac{\varepsilon}{\text{poly}(A)}\right)$, we have

$$I_1 \le I_2 + \varepsilon.$$

Handling the log-determinant term. Now consider the $\log|P|$ term. As we are decreasing each element by atmost $\beta\rho$, none of the eigenvalues can increase. Moreover, as

$$\text{Tr}(P') - \text{Tr}(P) \ge -d\beta\rho,$$

we can conclude that each eigenvalue decreases by at most $-d\beta\rho$. Also, as $\rho \le \kappa\lambda_j(P)\ \forall\ j \in [d]$, we can conclude that each eigenvalue satisfies

$$\lambda_j(P') \ge \lambda_j(P)(1 - d\beta\kappa).$$

Hence, the log-determinant satisfies

$$\log|P'| \ge \log|P| + d\log(1-\beta d\kappa) \ge \log|P| - \frac{d^2\beta\kappa}{1-d\beta\kappa} \ge \log|P| - O(\varepsilon) \text{ for } \beta \le \frac{\varepsilon}{\kappa d^2} \le \frac{\varepsilon}{\kappa r^2 d^2}.$$

This finally gives

$$|\gamma_{\theta,P} - \gamma_{\theta,P'}| \le O(\varepsilon).$$

Bounding the size of the net As $\beta = O\left(\frac{\varepsilon}{\text{poly}(A)}\right)$, and the max radius is also $O(\text{poly}(A))$, we have a cover of size $\left(\frac{\text{poly}(A)}{\varepsilon}\right)$ per entry of the precision matrix (for a fixed Ω_ρ).

Intersecting the d^2 nets means that for each Ω_ρ, we have a net of size

$$\left(\text{poly}\left(A, \frac{1}{\varepsilon}\right)\right)^{d^2}.$$

As we are considering $\text{poly}(A)$ many Ω_ρs, the size of the net remains the same as the above.

□

Claim C.13. *In the setting of Lemma C.5, if $P \in \Omega_\rho$ and $P' \in \widetilde{\Omega}_{\rho,\beta}$ is its nearest neighbor, then for $\gamma = 2\kappa\beta d^2$, we have*

$$(1-\gamma)\|t-\theta\|_\Sigma^2 \le \|t-\theta\|_{\Sigma'}^2 \le (1+\gamma)\|t-\theta\|_\Sigma^2, \tag{32}$$

where $\Sigma := P^{-1}, \Sigma' := P'^{-1}$.

Proof. Consider $P \in \Omega_\rho$ and $P' \in \widetilde{\Omega}_{\rho,\beta}$ such that $P = P' + \Delta$. Since P' is the rounding down of P, we have $\Delta_{ij} \in [0, \beta\rho]$.

As $\text{Tr}(\Delta) \in [0, \rho\beta d]$, and $\|\Delta\|_F \le \rho\beta d$, we have

$$\lambda_{\max}(\Delta) \le \rho\beta d \text{ and } \lambda_{\min}(\Delta) \ge -\rho\beta d^2.$$

This implies that when considering untruncated Gaussians with precision matrices P, P', we have that for all $t, \theta \in \mathbb{R}^d$,

$$\|t-\theta\|_\Sigma^2 - \rho\beta d^2\|t-\theta\|^2 \le \|t-\theta\|_{\Sigma'}^2 \le \|t-\theta\|_\Sigma^2 + \rho\beta d\|t-\theta\|^2.$$

Since $\lambda_{\min}(P) \ge \frac{\rho}{2\kappa}$, we have

$$\rho\|t-\theta\|^2 \le 2\kappa\|t-\theta\|_\Sigma^2.$$

Substituting in the previous inequality, we get

$$\begin{aligned}
&\|t-\theta\|_\Sigma^2 - \rho\beta d^2\|t-\theta\|^2 \le \|t-\theta\|_{\Sigma'}^2 \le \|t-\theta\|_\Sigma^2 + \rho\beta d\|t-\theta\|^2,\\
\implies\ &(1-2\kappa\beta d^2)\|t-\theta\|_\Sigma^2 \le \|t-\theta\|_{\Sigma'}^2 \le (1+2\kappa\beta d)\|t-\theta\|_\Sigma^2,
\end{aligned}$$

For the sake of symmetry, we will use the weaker bound of

$$(1-2\kappa\beta d^2)\|t-\theta\|_\Sigma^2 \le \|t-\theta\|_{\Sigma'}^2 \le (1+2\kappa\beta d^2)\|t-\theta\|_\Sigma^2,$$

Setting $\gamma = 2\kappa\beta d^2$ completes the proof. □

Lemma C.14. *Consider a bounded mean vector μ with $\|\mu\|_\infty \le \alpha$ and precision matrix P with max eigenvalue ρ and condition number κ.*

For $\gamma = O\Big(\min\Big\{\frac{\varepsilon}{\alpha\rho^{1/2}d^{3/2}}, \frac{\varepsilon}{\alpha^2 d^3\rho}\Big\}\Big)$, we have

$$\log\int_{t\le 0}\exp\left(-\frac{(1-\gamma)}{2}\|P^{\frac{1}{2}}(t-\mu)\|^2\right) \le \varepsilon + \log\int_{t\le 0}\exp\left(-\frac{1}{2}\|P^{\frac{1}{2}}(t-\mu)\|^2\right).$$

Proof of Lemma C.14. Wlog, consider $\mu \ge 0$. The case where the entries are possibly negative follow a similar proof.

Define the integral on the LHS and RHS of the Lemma statement by I_1 and I_2 respectively.

By a change of variables, we set $t' = \sqrt{1-\gamma}(t-\mu) + \mu$ in I_1, to get

$$I_1 = \log\frac{1}{\sqrt{1-\gamma}} + \log\int_{t'\le(1-\sqrt{1-\gamma})\mu}\exp\left(-\frac{1}{2}\|P^{\frac{1}{2}}(t'-\mu)\|^2\right).$$

Since $\gamma < 1$, we have $(1-\sqrt{1-\gamma})\mu < \gamma\mu$. Substituting in I_1, and for $\gamma = O(\varepsilon)$, we get

$$\begin{aligned}
I_1 &\le \log\frac{1}{\sqrt{1-\gamma}} + \log\int_{t'\le\gamma\mu}\exp\left(-\frac{1}{2}\|P^{\frac{1}{2}}(t'-\mu)\|^2\right),\\
&\le O(\varepsilon) + \log\int_{t'\le\gamma\mu}\exp\left(-\frac{1}{2}\|P^{\frac{1}{2}}(t'-\mu)\|^2\right).
\end{aligned}$$

The integrating set in the above inequality can be split into two parts: one over the negative orthant (which is exactly to e^{I_2}) and another over the shell

$$C = \{t' \le \gamma\mu\} \setminus \{t' \le 0\}.$$

This gives

$$I_1 \le O(\varepsilon) + \log\left(e^{I_2} + \int_{t'\in C}\exp\left(-\frac{1}{2}\|P^{\frac{1}{2}}(t'-\mu)\|^2\right)\right).$$

In the above inequality, let e^{I_3} denote the integral over the shell C. We will now show that I_3 satisfies

$$e^{I_3} \le \varepsilon e^{I_2}.$$

Let $f(x)$ denote the Gaussian density with mean μ and precision matrix P.

For a subset of co-ordinates $S \subseteq [d], S \ne \emptyset$, and $t \in \mathbb{R}^d$, let $x_+, x_- \in \mathbb{R}^d$ be such that

$$x_{+,S}(i) = \begin{cases}\gamma\mu_i & \text{if } i \in S,\\ t_i & \text{if } i \notin S,\end{cases} \quad , \quad x_{-,S}(i) = \begin{cases}-\frac{\gamma}{\varepsilon}\mu_i & \text{if } i \in S,\\ t_i & \text{if } i \notin S.\end{cases}$$

By the monotonicity of the Gaussian density, the integral over the shell C can be upper bounded by breaking up into a sum of integrals over lower-dimensional strips, where for a fixed subset $S \in [d]$, the variables t_{S^c} are integrated over $(-\infty, \gamma\mu_{S^c}]$, while the variables in S are fixed to $\gamma\mu_S$.

This gives

$$\begin{aligned}
e^{I_3} &\leq \sum_{S\subseteq[d]} \int_{t_{S^c}\leq\gamma\mu_{S^c}} f(x_{+,S}) \prod_{i\in S} \gamma\mu_i, \\
&\leq \sum_{S\subseteq[d]} \int_{t_{S^c}\leq\gamma\mu_{S^c}} f(x_{+,S})(\gamma\alpha)^{|S|}, \\
&\leq \sum_{k=1}^{d} \binom{d}{k} (\gamma\alpha)^k \max_{S\subseteq[d]:|S|=k} \int_{t_{S^c}\leq\gamma\mu_{S^c}} f(x_{+,S}).
\end{aligned}$$

By Claim C.15, for any S, and $\gamma = O\Big(\min\Big\{\frac{\varepsilon}{\alpha\sqrt{\rho d}}, \frac{\varepsilon}{\alpha^2 d^2 \rho}\Big\}\Big)$ each summand satisfies

$$f(x_{+,S}) \leq 2f(x_{-,S})$$

Furthermore, for any $S \subseteq [d]$, we have

$$\int_{t_{S^c}\leq\gamma\mu_{S^c}} f(x_{-,S})\Big(\frac{\gamma}{\varepsilon}\alpha\Big)^{|S|} \leq e^{I_2}.$$

This gives

$$e^{I_3} \leq \sum_{k=1}^{d} d^k 2\varepsilon^k e^{I_2} \leq 3\varepsilon d e^{I_2} \quad \text{if } \varepsilon d \leq \frac{1}{3}.$$

Rescaling $\varepsilon \leftarrow \frac{\varepsilon}{3d}$ completes the proof. □

Claim C.15. *Let f be the Gaussian density with mean $\mu \in [0,\alpha]^d$ and precision matrix $P \in \mathbb{R}^{d\times d}$ with max eigenvalue ρ and condition number κ.*

Let $\gamma = O\Big(\min\Big\{\frac{\varepsilon}{\alpha\sqrt{\rho d}}, \frac{\varepsilon}{\alpha^2 d^2 \rho}\Big\}\Big)$. For any subset of co-ordinates $S \subseteq [d], S \neq \emptyset$, and $t \in \mathbb{R}^d$, let $x_+, x_- \in \mathbb{R}^d$ be such that

$$x_+(i) = \begin{cases} \gamma\mu_i & \text{if } i \in S, \\ t_i & \text{if } i \notin S, \end{cases} \quad , \quad x_-(i) = \begin{cases} -\frac{\gamma}{\varepsilon}\mu_i & \text{if } i \in S, \\ t_i & \text{if } i \notin S. \end{cases}$$

we have

$$f(x_+) \leq 2f(x_-)$$

Proof. WLOG, let S be a contiguous set such that we can separate the coordinates of x_+ and x_- into disjoint sets. For the coordinates belonging to S, let μ_S denote the coordinates of μ belonging to μ, and μ_{S^c} the coordinates not belonging to S (similarly for t_S and t_{S^c}).

Taking the logarithm on both sides of the claimed inequality, we want to show that

$$-\frac{1}{2}\Big\|P^{\frac{1}{2}}\begin{bmatrix}\gamma\mu_S - \mu_S \\ t_{S^c} - \mu_{S^c}\end{bmatrix}\Big\|^2 \leq -\frac{1}{2}\Big\|P^{\frac{1}{2}}\begin{bmatrix}-\frac{\gamma}{\varepsilon}\mu_S - \mu_S \\ t_{S^c} - \mu_{S^c}\end{bmatrix}\Big\|^2 + \log 2$$

Let a and b denote the vectors whose norms correspond to the log-densities in the claimed inequality, and let $\delta = a - b$.

This gives

$$b = P^{\frac{1}{2}}\begin{bmatrix}-\mu_S(1-\gamma) \\ t_{S^c} - \mu_{S^c}\end{bmatrix}, \qquad a \quad = P^{\frac{1}{2}}\begin{bmatrix}-\mu_S(1+\frac{\gamma}{\varepsilon}) \\ t_{S^c} - \mu_{S^c}\end{bmatrix}, \qquad \delta := a - b = P^{\frac{1}{2}}\begin{bmatrix}-\mu_S\gamma\big(\frac{1}{\varepsilon}+1\big) \\ 0_{S^c}\end{bmatrix}$$

We want to show that

$$-\frac{1}{2}\|b\|^2 \leq -\frac{1}{2}\|a\|^2 + \log 2,$$
$$\Leftrightarrow \langle \delta, b \rangle + \frac{1}{2}\|\delta\|^2 \leq \log 2. \tag{33}$$

As $\|P\| \leq \rho$ and $\|\mu\|_\infty \leq \alpha$, we have

$$\|\delta\|_2^2 \leq \rho(\alpha^2|S|)\gamma^2\left(1 + \frac{1}{\varepsilon}\right)^2.$$

For $\gamma = O\left(\frac{\varepsilon}{\alpha\sqrt{\rho d}}\right)$, we get

$$\|\delta\|_2^2 \leq \frac{1}{2}\log 2. \tag{34}$$

Similarly, consider the inner product $\langle \delta, b \rangle$ in Eqn (33). By the trace trick, we get

$$\langle \delta, b \rangle = \mathrm{Tr}(\Delta P),$$

where

$$\Delta = \begin{bmatrix} -\mu_S(1-\gamma) \\ t_{S^c} - \mu_{S^c} \end{bmatrix} \begin{bmatrix} -\mu_S\gamma(\frac{1}{\varepsilon}+1) \\ 0_{S^c} \end{bmatrix}^T$$

Notice that the diagonal elements of Δ are all non-negative. This implies that all singular values are non-negative. The trace of Δ is

$$\mathrm{Tr}(\Delta) = \|\mu_S\|_2^2(1-\gamma)\gamma\left(\frac{1}{\varepsilon} + 1\right).$$

Hence, by Von Neumann's trace inequality, we get

$$\langle \delta, b \rangle \leq \mathrm{Tr}(\Delta)\mathrm{Tr}(P) \leq \|\mu_S\|_2^2(1-\gamma)\gamma\left(\frac{1}{\varepsilon} + 1\right)\rho d.$$

For $\gamma = O\left(\frac{\varepsilon}{\alpha^2|S|\rho d}\right)$, this gives

$$\langle \delta, b \rangle \leq \frac{1}{2}\log 2. \tag{35}$$

Substituting Eqn (34) and Eqn (35) in Eqn (33) completes the proof. □

Lemma C.6 (W-net)**.** *Let η_{S^c}, η_S be such that*

$$\|P_{S^c}^{*\frac{1}{2}}\eta_{S^c}\| \leq B_1, \|P_S^{*\frac{1}{2}}\eta_S\| \leq B_2,$$

for $B_1, B_2 \geq 0$.

Let $A > \max\{B_1^2, B_2^2, \mathrm{poly}(C,\kappa)\}$. Let $P^* = \Sigma^{*-1}$ be the precision matrix of η. For a fixed matrix $P \in \mathbb{R}^{d\times d}$ whose condition number satisfies Assumption 4.4 and whose eigenvalues satisfy $\lambda_{\max}(P) \in [e^{-\frac{2A}{d}}\lambda_{\min}(P^*), C\lambda_{\max}(P^*)]$, there exists a partition $\mathcal{I}$ of $\mathbb{R}^d$ with size

$$\left(\mathrm{poly}\left(A, \frac{1}{\varepsilon}\right)\right)^{3d}$$

such that for each interval $I \in \mathcal{I}$, we have one of the following:

- *for all $\theta \in I$, $\gamma_{\theta,P}(y) < -A$, or*
- *for all $\theta, \theta' \in I$, $|\gamma_{\theta,P}(y) - \gamma_{\theta',P}(y)| \leq \epsilon$.*

Proof of Lemma C.6. Recall that the log-likelihood ratio γ_θ can be decomposed into the difference of two terms that depend on θ:

$$\gamma_{\theta,P}(y) - \frac{1}{2}\log\frac{|P|}{|P'|} = g_{\theta,P}(y) - g_{\theta^*,P^*}(y) + h_{\theta,P}(y) - h_{\theta^*,P^*}(y).$$

Without loss of generality, consider the net for the first coordinate θ_1. The final net will be the intersection of the per-coordinate nets.

We will construct three partitions: the first is $\mathcal{I}_{h,0}$ for h when $y_1 = 0$, the second is is $\mathcal{I}_{h,1}$ for h when $y_1 > 0$, and the last is $\mathcal{I}_g$ for g when $y_1 > 0$. The final partition will be the intersection of these partitions.

Case 1: Net over h, $y_1 = 0$. As $y_1 = 0$, we have $1 \in S$. For $\theta \in \mathbb{R}^d$, we have

$$h_{\theta,P}(y) = \log \int_{t \le 0} \exp\Big(-\|P_S^{\frac{1}{2}}(t - \theta_S) + (P_S)^{-1/2} P_{SS^c}(y_{S^c} - \theta_{S^c})\|^2/2\Big),$$

By Claim C.16, if $\theta_1 \ge \Theta(\frac{\sqrt{C\kappa A}}{p_1})$, then the log-likelihood is smaller than $-A$.

Now, consider $\theta_1 < O(\frac{\sqrt{C\kappa A}}{p_1})$. Let $\theta' = \theta + \alpha e_1$ for $\alpha > 0$. As h is monotonically decreasing per coordinate, $\theta_1 < \theta'_1 \implies h_{\theta,P} \ge h_{\theta',P}$. We would like to now upper bound $h_{\theta,P}$ in terms of $h_{\theta',P}$.

Let

$$\begin{aligned}
\mu &:= \theta_S + (P_S)^{-1} P_{SS^c}(y_{S^c} - \theta_{S^c}),\\
\mu' &:= \theta'_S + (P_S)^{-1} P_{SS^c}(y_{S^c} - \theta_{S^c})
\end{aligned}$$

In the function h_θ, break the integrating set into two domains: one where $t - \mu$ is small,

$$\Omega_1 = \Big\{ t \in \mathbb{R}^{|S|} : \|P_S^{\frac{1}{2}}(t - \mu)\| \le r \Big\},$$

and another where it is large:

$$\Omega_2 = \Big\{ t \in \mathbb{R}^{|S|} : \|P_S^{\frac{1}{2}}(t - \mu)\| > r \Big\},$$

for some $r > 0$ that we will specify later.

Let I_1 and I_2 denote the integrals over Ω_1 and Ω_2 respectively.

I_2 corresponds to the tail of an unnormalized Gaussian distribution, and hence we have

$$\begin{aligned}
&h_{\theta,P}(y) = \log\left(I_2 + \int_{t \le 0, t \in \Omega_1} \exp\Big(-\|P_S^{\frac{1}{2}}(t - \mu)\|^2/2\Big)\right),\\
&\text{where } I_2 \le (2\pi)^{\frac{|S|}{2}} \left|P_S^{-\frac{1}{2}}\right| e^{-r^2}.
\end{aligned}$$

We can simplify I_2 be comparing $|P|$ to $|P^*|$:

$$\begin{aligned}
I_2 &\le (2\pi)^{\frac{|S|}{2}} \frac{\left|P_S^{-1/2}\right|}{\left|P_S^{*-1/2}\right|} \left|P_S^{*-1/2}\right| e^{-r^2} \le (2\pi)^{\frac{|S|}{2}} \left(\frac{\lambda^*_{\max}}{\lambda_{\min}(P)}\right)^{|S|} \left|P_S^{*-\frac{1}{2}}\right| e^{-r^2},\\
&\le (2\pi)^{\frac{|S|}{2}} \left(\kappa e^{\frac{A}{d}}\right)^{|S|} \left|P_S^{*-\frac{1}{2}}\right| e^{-r^2}.
\end{aligned}$$

By Lemma C.10, we have

$$(2\pi)^{\frac{|S|}{2}} \left|P_S^{*-\frac{1}{2}}\right| \le e^{h_{\theta^*,P^*}(y) + O(d + B_2^2 + B_3^2)}$$

As we are only consider θ' such that $h_{\theta^*,P^*}(y) - A < h_{\theta',P}(y)$, for

$$r^2 = O(d \log \kappa + A + \log \frac{1}{\varepsilon}) = O(A),$$

we have

$$I_2 \leq \varepsilon e^{h_{\theta',P}(y)}$$

Subsituting in $h_{\theta,P}(y)$, we get

$$h_{\theta,P}(y) \leq \log\left(\varepsilon e^{h_{\theta',P}(y)} + \int_{t \leq 0, t \in \Omega_1} \exp\left(-\|P_S^{\frac{1}{2}}(t-\mu)\|^2/2\right)\right)$$

Now consider the integral $I_1 = \int_{t \leq 0, t \in \Omega_1} \exp\left(-\|P_S^{\frac{1}{2}}(t-\mu)\|^2/2\right)$.

By Claim C.17, as Ω_1 is defined for t bounded by r, and $\mu - \mu' = \alpha e_1$, we have

$$I_1 \leq \exp\left(2\alpha p_1 r + \alpha^2 p_1^2\right) \cdot I_1',$$
$$\text{where } I_1' = \int_{t \leq 0, t \in \Omega_1} \exp\left(-\|P_S^{\frac{1}{2}}(t-\mu')\|^2/2\right) \leq e^{h_{\theta',P}(y)}.$$

Substituting in the expression for $h_{\theta,P}$, we get

$$h_{\theta,P}(y) \leq \log\left(e^{h_{\theta',P}(y)} \cdot \left(\varepsilon + \exp\left(2\alpha p_1 r + \alpha^2 p_1^2\right)\right)\right)$$

As $\log(\varepsilon + e^x) \leq \varepsilon + x$ for $x \geq 0$, we have

$$h_{\theta,P}(y) \leq h_{\theta',P}(y) + \varepsilon + 2\alpha p_1 r + \alpha^2 p_1^2.$$

Setting $\alpha = O(\frac{\varepsilon}{p_1 r})$, we get

$$h_{\theta,P}(y) \leq h_{\theta',P}(y) + 2\varepsilon.$$

This shows that $h_{\theta,P}$ changes by at most ε for the considered net. We need to defined the other end point for the net. By a similar argument to the positive end point, if $\theta_1 = -O\left(\frac{\sqrt{C\kappa \log\left(\frac{1}{\varepsilon}\right)}}{p_1}\right)$, the log-likelihood ratio changes by at most ε until $\theta_1 = -\infty$.

As we are only trying to cover θ such that $|\theta_1| \leq O(\frac{\sqrt{C\kappa A}}{p_1})$, this net has size

$$O\left(\frac{\sqrt{C\kappa A}}{p_1 \alpha}\right) = O\left(\frac{\sqrt{C\kappa A}}{p_1}\frac{p_1 r}{\varepsilon}\right) = \frac{A}{\varepsilon}.$$

Case 2: Net over h, $y > 0$. A similar argument to Case 1 works here as well.

Case 3: Net over g, $y_1 > 0$. By Lemma C.11, if $|\theta - \theta_*|_{P_{S^c}} > R_1$ for

$$R_1 = O(\sqrt{A}),$$

then

$$g_{\theta,P} - g_{\theta^*,P} < -A.$$

Now consider θ such that

$$|\theta_1 - \theta_1^*| \leq R,$$

and θ, θ' such that $\theta - \theta' = \alpha e_1$.

The difference in $g_\theta - g_{\theta'}$ is

$$g_\theta(y) - g_{\theta'}(y) = \eta_{S^c}^T(\Sigma_{S^c})^{-1}(\theta_{S^c} - \theta'_{S^c}) - \frac{1}{2}\|\theta^*_{S^c} - \theta_{S^c}\|^2_{\Sigma_{S^c}} + \frac{1}{2}\|\theta^*_{S^c} - \theta'_{S^c}\|^2_{\Sigma_{S^c}},$$

The second and third terms in the RHS can bounded by observing that

$$|2\theta^* - \theta' - \theta| \leq 2R_1 = O(\sqrt{A}), |\theta' - \theta| \leq \alpha,$$

and hence we get

$$-\frac{1}{2}\|\theta^*_{S^c} - \theta_{S^c}\|^2_{\Sigma_{S^c}} + \frac{1}{2}\|\theta^*_{S^c} - \theta'_{S^c}\|^2_{\Sigma_{S^c}} \leq O(\sqrt{A})\alpha.$$

Now, for the first term in the RHS, we have

$$\|\eta_{S^c}\|_{P^*_{S^c}} \leq B_1 \implies \|\eta_{S^c}\| \leq \frac{B_1}{\lambda_{\min}^{\frac{1}{2}}(P^*)} \leq \frac{B_1\sqrt{\kappa}}{\lambda_{\max}^{\frac{1}{2}}(P^*)}.$$

This further implies that

$$\eta_{S^c}^T(\Sigma_{S^c})^{-1}(\theta_{S^c} - \theta'_{S^c}) \leq \frac{B_1\sqrt{\kappa}}{\lambda_{\max}^{\frac{1}{2}}(P^*)}\sqrt{p_1}\alpha = \mathrm{poly}(A)\alpha.$$

Setting

$$\alpha = O\left(\frac{\epsilon}{\mathrm{poly}(A)}\right),$$

we get

$$|g_\theta(y) - g_{\theta'}(y)| \leq \frac{\epsilon}{d}.$$

As we are covering a set of size R_1 using a grid size of α, the size of this partition is

$$O(\frac{R_1}{\alpha}) = \frac{\mathrm{poly}(A)}{\varepsilon}.$$

$\square$

Claim C.16. *In the setting of Lemma C.6, we have $\lambda_{\max}(P) \leq C\lambda_{\max}(P^*)$. Let p_1 denote the first diagonal element of P.*

If $\theta_1 \geq \Theta(\frac{\sqrt{C\kappa A}}{p_1})$, then the function $h_{\theta,P}$ is such that

$$h_{\theta,P} - h_{\theta^*,P} < -A.$$

Proof of Claim C.16. Recall that the function $h_{\theta,P}$ is defined as:

$$h_{\theta,P}(y) = \log\int_{t\leq 0}\exp\left(-\|P_S^{\frac{1}{2}}(t-\theta_S) + (P_S)^{-1/2}P_{SS^c}(y_{S^c} - \theta_{S^c})\|^2/2\right).$$

Consider the term $\|P_S^{\frac{1}{2}}(t-\theta_S) + (P_S)^{-1/2}P_{SS^c}(y_{S^c} - \theta_{S^c})\|$. By the triangle inequality, we have

$$\begin{aligned}
&\|P_S^{\frac{1}{2}}(t-\theta_S) + (P_S)^{-1/2}P_{SS^c}(y_{S^c} - \theta_{S^c})\| \\
&\quad\geq \|P_S^{\frac{1}{2}}(t-\theta_S) + (P_S)^{-1/2}P_{SS^c}(\theta'_{S^c} - \theta^*_{S^c})\| - \|P_S^{-\frac{1}{2}}P_{SS^c}(y_{S^c} - \theta^*_{S^c})\|, \\
&\quad\geq \|P_S^{\frac{1}{2}}(t-\theta_S) + (P_S)^{-1/2}P_{SS^c}(\theta'_{S^c} - \theta^*_{S^c})\| - \sqrt{C\kappa A},
\end{aligned}$$

where the last inequality follows as

$$\|P_S^{-\frac{1}{2}} P_{SS^c}(y_{S^c} - \theta^*_{S^c})\| = \|P_S^{-\frac{1}{2}} P_{SS^c}(\eta_{S^c})\| \le \|P_{S^c}^{\frac{1}{2}}\| \|\eta_{S^c}\| \le \frac{\sqrt{C\lambda^*_{\max}}}{\sqrt{\lambda^*_{\min}}} B \le \sqrt{C\kappa A}.$$

Similarly, the function $g_{\theta,P}$ only considers θ such that $\|P_{S^c}^{\frac{1}{2}}(\theta_{S^c} - \theta^*_{S^c})\| \le \sqrt{C\kappa A}$ (otherwise, the log-likelihood ratio is smaller than $-A$ by virtue of $g_{\theta,P}$, irrespective of $h_{\theta,P}$). For these θ, we have

$$\begin{aligned}&\|P_S^{\frac{1}{2}}(t - \theta_S) + (P_S)^{-1/2} P_{SS^c}(\theta^*_{S^c} - \theta_{S^c})\| \\ \ge&\|P_S^{\frac{1}{2}}(t - \theta_S)\| - \sqrt{C\kappa A},\end{aligned}$$

which gives

$$\|P_S^{\frac{1}{2}}(t - \theta_S) + (P_S)^{-1/2} P_{SS^c}(y_{S^c} - \theta_{S^c})\| \ge \|P_S^{\frac{1}{2}}(t - \theta_S)\| - 2\sqrt{C\kappa A}.$$

Hence, if $\theta_1 \ge O(\frac{\sqrt{C\kappa A}}{p_1})$, then we have

$$\|P_S^{\frac{1}{2}}(t - \theta_S) + (P_S)^{-1/2} P_{SS^c}(y_{S^c} - \theta_{S^c})\| \ge \Omega(\sqrt{C\kappa A}) \;\forall\, t \le 0,$$

and hence the Gaussian integral is at most $(2\pi)^{|S|/2}\left|P_S^{-\frac{1}{2}}\right| e^{-\Omega(C\kappa A)}$.

By Lemma C.10, we have $h_{\theta^*,P^*} \ge -\frac{1}{2}\log|P^*_S| - O(A)$, which gives

$$\begin{aligned}h_{\theta,P}(y) - h_{\theta^*,P^*}(y) &< \frac{1}{2}\log\frac{|P^*_S|}{|P_S|} - \Omega(C\kappa A) \\ &< \frac{d}{2}\log\frac{\lambda^*_{\max}}{\lambda_{\min}(P)} - \Omega(C\kappa A) < O(A\log\kappa) - \Omega(C\kappa A) = -\Omega(C\kappa A).\end{aligned}$$

This gives a contiguous interval over θ_1 for which $\gamma_{\theta,P} < -A$. □

Claim C.17. *In the setting of Lemma C.6, let μ, μ' be such that $\mu - \mu' = \alpha e_1$*

Then, for all t such that

$$\|P_S^{\frac{1}{2}}(t - \mu)\| \le r,$$

and $p_1 := P_{11}$, we have

$$\|P_S^{\frac{1}{2}}(t - \mu)\|^2 \ge -2\alpha p_1 r - \alpha^2 p_1^2 + \|P_S^{\frac{1}{2}}(t - \mu')\|^2.$$

Proof of Claim C.17. Consider the term $\|P_S^{\frac{1}{2}}(t - \mu)\|^2$.

Adding and subtracting $\|P_S^{\frac{1}{2}}(t - \mu')\|^2$, we get

$$\begin{aligned}\|P_S^{\frac{1}{2}}(t - \mu)\|^2 &= \|P_S^{\frac{1}{2}}(t - \mu)\|^2 - \|P_S^{\frac{1}{2}}(t - \mu')\|^2 + \|P_S^{\frac{1}{2}}(t - \mu')\|^2, \\ &= \langle P_S^{\frac{1}{2}}(2t - \mu' - \mu), P_S^{\frac{1}{2}}(\mu' - \mu)\rangle + \|P_S^{\frac{1}{2}}(t - \mu')\|^2, \\ &= \langle P_S^{\frac{1}{2}}(2t - 2\mu - (\mu' - \mu)), P_S^{\frac{1}{2}}(\mu' - \mu)\rangle + \|P_S^{\frac{1}{2}}(t - \mu')\|^2, \\ &= 2\langle P_S^{\frac{1}{2}}(t - \mu), P_S^{\frac{1}{2}}(\mu' - \mu)\rangle - \|P_S^{\frac{1}{2}}(\mu - \mu')\|^2 + \|P_S^{\frac{1}{2}}(t - \mu')\|^2.\end{aligned}$$

As $\mu - \mu' = \alpha e_1$, we have

$$\|P_S^{\frac{1}{2}}(\mu - \mu')\|^2 = \alpha^2 p_1^2.$$

By the Cauchy-Schwartz inequality, and since $\mu - \mu' = \alpha e_1$, the inner product can be lower bounded as

$$2\langle P_S^{\frac{1}{2}}(t - \mu), P_S^{\frac{1}{2}}(\mu' - \mu)\rangle \ge -2\|P_S^{\frac{1}{2}}(t - \mu)\| \|P_S^{\frac{1}{2}}(\mu' - \mu)\| \ge -2\alpha p_1 r.$$

Substituting, we get

$$\|P_S^{\frac{1}{2}}(t - \mu)\|^2 \ge -2\alpha p_1 r - \alpha^2 p_1^2 + \|P_S^{\frac{1}{2}}(t - \mu')\|^2.$$

This completes the proof.

□

Lemma C.18. *Following Definition C.12 let Ω_ρ be the set of precision matrices with condition number κ satisfying $\lambda_{\max}(P) \in [\frac{\rho}{2}, \rho]$, and let $\widetilde{\Omega}_{\rho,\beta}$ be the quantized net with quantization level β. For any $P \in \Omega_\rho$, let $\widetilde{P} \in \widetilde{\Omega}_{\rho,\beta}$ be its element-wise rounding down. Then, for any $\mu \in \mathbb{R}^d$, we have*

$$d_{TV}(\mathcal{N}(\mu; P), \mathcal{N}(\mu; \widetilde{P})) \leq O\big(d^2 \beta\kappa\big). \tag{36}$$

Proof of Lemma C.18. Let $\Sigma = P^{-1}$ and $\widetilde{\Sigma} = \widetilde{P}^{-1}$.

By Theorem 1.1 in [16], the TV between two Gaussians with the same mean is

$$d_{TV}(\mathcal{N}(\mu; \Sigma), \mathcal{N}(\mu; \widetilde{\Sigma})) = \Theta\left(\min\left\{1, \sqrt{\sum_i \xi_i^2}\right\}\right),$$

where ξ_i are the eigenvalues of $\widetilde{\Sigma}^{-1}\Sigma - I_d$.

We can convert the bound on the eigenvalues to the Frobenius norm of $\widetilde{\Sigma}^{-1}\Sigma - I_d$:

$$\sqrt{\sum_i \xi_i^2} \leq \|\widetilde{\Sigma}^{-1}\Sigma - I_d\|_F.$$

Recall that $\widetilde{P}$ is the rounding down per entry of P. Hence,

$$\begin{aligned}\widetilde{\Sigma}^{-1}\Sigma - I_d &= (\Sigma^{-1} - [\nu_{ij}])\Sigma - I_d, \text{ where } 0 \leq \nu_{ij} < \beta\rho,\\ &= -\nu\Sigma.\end{aligned}$$

Taking the Frobenius norm, we get

$$\begin{aligned}\|\widetilde{\Sigma}^{-1}\Sigma - I_d\|_F &= \|\nu\Sigma\|_F,\\ &\leq (d\beta\rho)(d\rho_{\max}(\Sigma)) = (d\beta\rho)\left(\frac{d}{\rho_{\min}(P)}\right),\\ &\leq (d\beta\rho)\left(\frac{d\kappa}{\rho_{\max}(P)}\right) \leq 2d^2\beta\kappa.\end{aligned}$$

where the first inequality follows as each element of ν is at most $\beta\rho$, the second inequality follows as P has condition number κ, and the third follows as $P \in \Omega_\rho \implies \rho_{\max}(P) \geq \frac{\rho}{2}$.

This completes the proof. $\square$

Lemma C.7. *Let $x_1, \dots, x_n$ be fixed, and $y_i = \phi(W^* x_i + \eta_i)$ for $\eta_i \sim \mathcal{N}(0, \Sigma^*)$, and $W^* \in \mathbb{R}^{d\times k}$ with $\Sigma^* \in \mathbb{R}^{d\times d}$ satisfying Assumption 4.4 and Assumption C.3. For a sufficiently large constant $C > 0$,*

$$n = C \cdot \frac{(d^2 + kd)}{\varepsilon^2} \log \frac{kd\kappa}{\varepsilon}$$

samples suffice to guarantee that with high probability, the MLE $\widehat{W}, \widehat{\Sigma}$ satisfies

$$\widetilde{d}\Big((\widehat{W}, \widehat{\Sigma}), (W^*, \Sigma^*)\Big) \leq \varepsilon.$$

Proof of Lemma C.7. For any $W \in \mathbb{R}^{d\times k}$, $\Sigma \in \mathbb{R}^{d\times d}$ and a sample (x_i, y_i), let $p_{i,W,\Sigma}(y|x_i)$ be the conditional distribution of $y = \phi(Wx + \eta)$, and let $\gamma_{i,W,\Sigma}$ be the log-likelihood ratio between (W, Σ) and (W^*, Σ^*) on this sample:

$$\gamma_{i,W,\Sigma}(y) := \log \frac{p_{i,W,\Sigma}(y \mid x_i)}{p_{i,W^*,\Sigma^*}(y \mid x_i)}.$$

Then

$$\mathop{\mathbb{E}}_y[\gamma_{i,W,\Sigma}(y)] = -KL(p_{i,W^*,\Sigma^*}(y \mid x_i) \| p_{i,W,\Sigma}(y \mid x_i)).$$

Concentration. From Lemma B.1, we see that if $d_{TV}((W^*, \Sigma^*), (W, \Sigma)) \geq \varepsilon$, then for $n \geq O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta})$,

$$\overline{\gamma}_{W,\Sigma} := \frac{1}{n} \sum_{i=1}^{n} \gamma_{i,W,\Sigma}(y_i) < -\frac{\varepsilon^2}{2}, \tag{37}$$

with probability $1 - \delta$.

Of course, whenever $\overline{\gamma}_{W,\Sigma} < 0$, the likelihood under W^*, Σ^* is larger than the likelihood under W, Σ. Thus, for each *fixed* W, Σ with $d_{TV}((W^*, \Sigma^*), (W, \Sigma)) \geq \varepsilon$, maximizing likelihood would prefer W^*, Σ^* to W, Σ with probability $1 - \delta$ if $n \geq O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta})$.

Nothing above is specific to our ReLU-based distribution. But to extend to the MLE over all W, Σ, we need to build a net using properties of our distribution.

Building a net. First, for a given sample y_i, let S_i be the set of coordinates of y_i that are zero, and S_i^c be its complement. Then, with high probability,

$$\|P_{S_i}^{*\frac{1}{2}}(\eta_i)_{S_i}\|, \|P_{S_i^c}^{*\frac{1}{2}}(\eta_i)_{S_i}\| \leq B = O(\sqrt{\kappa d \log n}) \; \forall \; i \in [n],$$

where $P^* = \Sigma^{*-1}$, and $P^*_S, P^*_{S^c}$ are the block matrices in P^*.

Supposing the above event happens, we will construct a net over the precision matrices $P = \Sigma^{-1}$. Note that as we are only considering matrices with bounded condition number, this is a bijective mapping.

Net over Σ^{-1}. By Lemma C.4, any precision matrix $P = \Sigma^{-1}$ satisfying Assumption 4.4 and whose max eigenvalue satisfies

$$\lambda_{\max}(P) \geq U \cdot \lambda_{\max}(P^*),$$

for $U = O\left(\frac{\kappa^3 d^2 n^2}{k^2} + \frac{\kappa^2 dn \log n}{k}\right)$, will have $\bar{\gamma}_{W,P^{-1}} < 0$, irrespective of W.

Similarly, by Lemma C.2, any precision matrix satisfying Assumption 4.4 and whose max eigenvalue satisfies

$$\lambda_{\max}(P) \leq L \cdot \lambda_{\min}(P^*)$$

for $L = e^{-O(\kappa \log n)}$ has $\gamma_{i,W,P^{-1}} \leq 0$ for all $i \in [n]$, and hence its average $\bar{\gamma}_{W,P^{-1}}$ is also < 0.

This shows that for all precision matrices P whose max eigenvalue is extremely small / large when compared to the min / max eigenvalues of P^*, has

$$\bar{\gamma}_{W,P^{-1}} < 0,$$

irrespective of W, and the MLE, which has non-negative $\bar{\gamma}$, will never pick these P.

Let $A = \text{poly}(n, d, \kappa, \frac{1}{\varepsilon})$ be large enough such that $A > n(d \log(U\kappa) + B^2)$, and such that it meets the requirements of Lemma C.5 and Lemma C.6.

Then, by Lemma C.5 with $U = \text{poly}(d, \kappa, n)$ and $\log(\frac{1}{L}) = \text{poly}(\kappa, n)$, there exists a partition $\mathcal{P}$ of precision matrices whose max-eigenvalue lies in $[L \cdot \lambda_{\max}(P^*), U \cdot \lambda_{\max}(P^*)]$ into $\left(\text{poly}\left(d, \kappa, n, \frac{1}{\varepsilon}\right)\right)^{d^2}$ cells, such that for each cell $I \in \mathcal{P}$, and $P, P' \in I$, the following holds for all $i \in [n]$ and $W \in \mathbb{R}^{d \times k}$:

$$|\gamma_{i,W,P}(y_i) - \gamma_{i,W,P'}(y_i)| \leq \frac{\varepsilon^2}{16} \tag{38}$$

or $\gamma_{i,W,P}(y_i) < -A$.

Using Lemma C.18, we also have that for all W,

$$d_{TV}((W, P), (W, P')) \leq \frac{\varepsilon^2}{16}. \tag{39}$$

We can choose a net N consisting of precision matrices from each cell in $\mathcal{P}$. This net has size

$$\log|N| \lesssim d^2 \log\left(\frac{d\kappa n}{\varepsilon}\right).$$

This gives a sufficient net over the precision matrices.

Now we will construct a net over W for each precision matrix in the net.

W-net. Now, for each $\widetilde{P} \in N_P$, by Lemma C.6, for each $i \in [n]$, there exists a partition $\mathcal{P}_{\widetilde{P},i}$ of $\mathbb{R}^d$ into $\left(\operatorname{poly}\left(d, k, \kappa, n, \frac{1}{\varepsilon}\right)\right)^d$ cells such that for each cell $I \in \mathcal{P}_{\widetilde{P},i}$, and $W, W' \in I$, one of the following holds:

$$\left|\gamma_{i,W,\widetilde{P}}(y_i) - \gamma_{i,W',\widetilde{P}}(y_i)\right| \leq \frac{\varepsilon^2}{16} \tag{40}$$

or $\gamma_{i,W,\widetilde{P}}(y_i) < -A$.

Let W_j be the j-th row of W. The individual partitions $\mathcal{P}_{\widetilde{P},i}$ on $\langle x_i, W_j\rangle$ induce a partition $\mathcal{P}_{\widetilde{P},i,j}$ on $\mathbb{R}^k$, where W_j, W_j' lie in the same cell of $\mathcal{P}_{\widetilde{P},i,j}$ if $\langle x_i, W_j\rangle$ and $\langle x_i, W_j'\rangle$ are in the same cell of $\mathcal{P}_{\widetilde{P},i}$ for all $i \in [n]$. Since $\mathcal{P}_{\widetilde{P},i,j}$ is defined by n sets of $\left(\operatorname{poly}\left(d, k, \kappa, n, \frac{1}{\varepsilon}\right)\right)$ parallel hyperplanes in $\mathbb{R}^k$, the number of cells in $\mathcal{P}_{\widetilde{P},i,j}$ is

$$\left(\operatorname{poly}\left(d, k, \kappa, n, \frac{1}{\varepsilon}\right)\right)^k.$$

As there are d rows in W, we can intersect $\mathcal{P}_{\widetilde{P},i,j}$ over $j \in [d]$, which induces $\operatorname{poly}(d, k, \kappa, n, \frac{1}{\varepsilon})^{kd}$ cells in $\mathbb{R}^k$. We choose a net $N_{\widetilde{P}}$ to contain, for each cell in $\bigcap_{j\in[d]} \mathcal{P}_{\widetilde{P},i,j}$, the W in the cell maximizing $d_{TV}((W^*, P^*), (W, \widetilde{P}))$. This has size

$$\log\left|N_{\widetilde{P}}\right| \lesssim kd \log\left(\frac{\kappa dkn}{\varepsilon}\right).$$

Proving MLE works. By (37), for our $n \geq O\left(\frac{(kd+d^2)}{\varepsilon^2} \log \frac{kd\kappa}{\varepsilon}\right)$, we have with high probability that

$$\overline{\gamma}_{W,P} \leq -\frac{\varepsilon^2}{2},$$

for all $P \in N$ and for all $W \in N_{\widetilde{P}}$ with $d_{TV}((W^*, P^*), (W, P)) \geq \varepsilon$. Suppose that both this happens, and

$$\|P^{*\frac{1}{2}}_{S_i}(\eta_i)_{S_i}\|, \|P^{*\frac{1}{2}}_{S_i^c}(\eta_i)_{S_i}\| \leq B = O(\sqrt{\kappa d \log n})\ \forall\, i \in [n].$$

We claim that the MLE $\widehat{W}, \widehat{\Sigma}$ must have $d_{TV}((W^*, \Sigma^*), (\widehat{W}, \widehat{\Sigma})) < \frac{17}{16}\varepsilon$.

Consider any $W \in \mathbb{R}^{d\times k}$ and $P \in \mathbb{R}^{d\times d}$ with $d_{TV}((W^*, P^*), (W, P)) \geq \frac{17}{16}\varepsilon$. Using our net on precision matrices, we can find $\widetilde{P} \in N$ such that

$$d_{TV}((W^*, P^*), (W, \widetilde{P})) \geq d_{TV}((W^*, P^*), (W, P)) - d_{TV}((W, P), (W, \widetilde{P})).$$

Recall that we are only currently considering W, P such that $d_{TV}((W^*, P^*), (W, P)) \geq \frac{17}{16}\varepsilon$. By Eqn (39), we have $d_{TV}((W, P), (W, \widetilde{P})) \leq \frac{\varepsilon^2}{16}$, which gives

$$d_{TV}((W^*, P^*), (W, \widetilde{P})) \geq \varepsilon\frac{17}{16} - \frac{\varepsilon^2}{16} \geq \varepsilon.$$

Now, for this $\widetilde{P}$, we can find a $\widetilde{W} \in N_{\widetilde{P}}$, and by our choice of $N_{\widetilde{P}}$, we know that

$$d_{TV}((W^*, P^*), (\widetilde{W}, \widetilde{P})) \geq d_{TV}((W^*, P^*), (W, \widetilde{P})) \geq \varepsilon,$$

and by (37), we have $\overline{\gamma}_{\widetilde{W},\widetilde{P}} \leq -\frac{\varepsilon^2}{2}$.

Now we consider two cases. In the first case, there exists i with $\gamma_{i,W,P}(y_i) < -A$. Then

$$\overline{\gamma}_{W,P} = \frac{1}{n}\sum_i \gamma_{i,W,P}(y_i) \leq -\frac{A}{n} + B^2/2 < 0.$$

Otherwise, by Eqn (38) and Eqn (40), we have

$$\begin{aligned}
\overline{\gamma}_{W,P} &\leq \overline{\gamma}_{\widetilde{W},\widetilde{P}} + \left|\overline{\gamma}_{\widetilde{W},\widetilde{P}} - \overline{\gamma}_{W,\widetilde{P}}\right| + \left|\overline{\gamma}_{W,\widetilde{P}} - \overline{\gamma}_{W,P}\right|, \\
&\leq -\frac{\varepsilon^2}{2} + \max_i\left|\overline{\gamma}_{i,\widetilde{W},\widetilde{P}} - \overline{\gamma}_{i,W,\widetilde{P}}\right| + \max_i\left|\overline{\gamma}_{i,W,\widetilde{P}} - \overline{\gamma}_{i,W,P}\right|, \\
&\leq -\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{16} + \frac{\varepsilon^2}{16} < 0.
\end{aligned}$$

In either case, $\overline{\gamma}_{W,P} < 0$ and the likelihood under w^* exceeds that under w. Hence the MLE $\widehat{w}$ must have $d_{TV}(w^*, w) \leq \frac{17}{16}\varepsilon$. Rescaling ε gives the conclusion of the Lemma.

□

Lemma C.8. *Let $\{x_i\}_{i=1}^n$ be i.i.d. random variables such that $x_i \sim \mathcal{D}_x$.*

Let $P^* := \Sigma^{*-1}$. Let $\lambda^*_{\min}, \lambda^*_{\max}$ be the minimum and maximum eigenvalues of P^*. For $0 < L < U$, let Ω denote the following set of precision matrices

$$\Omega := \left\{P \in \mathbb{R}_+^{d\times d} : \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \leq \kappa \text{ and } \lambda_{\max}(P) \in [L\cdot\lambda^*_{\min}, U\cdot\lambda^*_{\max}]\right\}.$$

Then, for a sufficiently large constant $C > 0$, and for

$$n = C\cdot\left(\frac{kd + d^2}{\varepsilon^2}\right)\log\left(\frac{kd\kappa}{\varepsilon}\log\left(\frac{U}{L}\right)\right),$$

we have:

$$\Pr_{x_i\sim\mathcal{D}_x}\left[\sup_{W\in\mathbb{R}^{d\times k}, P\in\Omega}\left|\widetilde{d}((W,P),(W^*,P^*)) - d((W,P),(W^*,P^*))\right| > \varepsilon\right] \leq e^{-\Omega(n\varepsilon^2)}.$$

Proof of Lemma C.8. For $P = \Sigma^{-1}$ and $P^* = \Sigma^{*-1}$, let

$$f(W,P) := d((W,\Sigma),(W^*,\Sigma^*))$$

and

$$f_n(W,P) := \widetilde{d}((W,\Sigma),(W^*,\Sigma^*)) = \frac{1}{n}\sum_i [d_{TV}(p_{W,P}(y|x_i), p_{W^*,P^*}(y|x_i))].$$

Since the function is bounded, for any fixed W, P, the Chernoff bound gives

$$\Pr[|f_n(W,P) - f(W,P)| > \alpha] \leq e^{-2n\alpha^2}. \tag{41}$$

for any $\alpha > 0$. The challenge lies in constructing a net to be able to union bound over $\mathbb{R}^k$ without assuming any bound on W or the covariate x. As before, we do so by constructing a "ghost" sample, symmetrizing, and constructing a net based on these samples.

Ghost sample. First, we construct a "ghost" dataset D'_x consisting of n fresh samples IID samples $\{x'_i\}_{i\in[n]}$ of $\mathcal{D}_x$. This gives another metric

$$f'_n(W,P) := \widetilde{d}'((W,\Sigma),(W^*,\Sigma^*)) = \frac{1}{n}\sum_i [d_{TV}(p_{W,P}(y|x'_i), p_{W^*,P^*}(y|x'_i))].$$

Similar to the proof in Lemma 4.3, it is sufficient to consider the difference between $f_n(W,P)$ and $f'_n(W,P)$ i.e.,

$$\Pr\left[\sup_{W,P}|f(W,P) - f_n(W,P)| > \varepsilon\right] \leq 2\Pr\left[\sup_{W,P}|f_n(W,P) - f'_n(W,P)| > \varepsilon/2\right]. \tag{42}$$

Symmetrization. Since D_x and D'_x each have n independent samples, we could instead draw the datasets by first sampling $2n$ elements $x_1, \ldots, x_{2n}$ from $\mathcal{D}_x$, then randomly partition this sample into two equal datasets. Let $s_i \in \{\pm 1\}$ so $s_i = 1$ if z_i lies in D'_x and -1 if it lies in D_x. Then

$$f_n(W, P) - f'_n(W, P) = \frac{1}{n} \sum_{i=1}^{2n} s_i \cdot d_{TV}(p_{W,P}(y|x_i), p_{W^*,P^*}(y|x_i)).$$

For a fixed W, P and $x_1, \ldots, x_{2n}$, the random variables $(s_1, \ldots, s_{2n})$ are a permutation distribution, so negatively associated. Then the variables $s_i \cdot d_{TV}(p_{W,P}(y|x_i), p_{W^*,P^*}(y|x_i))$ are monotone functions of s_i, so also negatively associated. They are also bounded in $[-1, 1]$. Hence we can apply a Chernoff bound:

$$\Pr[|f_n(W, P) - f'_n(W, P)| > \varepsilon] < e^{-n\varepsilon^2/2}, \tag{43}$$

for any fixed W, P.

Constructing a net. We will first construct a net over the precision matrices P (independent of W), and then for each element in the P-net, we will construct a net over W.

Net over Σ^{-1}. In the following, $\lambda_{\max}(P)$ denotes max eigenvalue of a matrix P, $\lambda_{\min}(P)$ denotes the min eigenvalue, and $\lambda_i(P)$ denotes the i-th eigenvalue, in decreasing order.

In order to construct the net over the precision matrices, we will consider geometrically spaced values of $\lambda \in [L \cdot \lambda_{\min}(P^*), U \cdot \lambda_{\max}(P^*)]$, and for each λ, we will construct a net over matrices that have max eigenvalue $\leq \lambda$.

Now consider $\lambda > 0$ that lies in the following discrete set:

$$\left\{\lambda_{\min}(P^*) 2^j, j \in \lceil \log_2(\kappa \tfrac{U}{L}) \rceil\right\}$$

This set is a geometric partition over the possible max eigenvalues that the MLE can return.

Following definition C.12, let Ω_λ denote the subset of positive definite matrices in $\mathbb{R}^{d \times d}$ that have condition number κ and max-eigenvalue in $\left[\frac{\lambda}{2}, \lambda\right]$. Similarly, following Definition C.12, let $\widetilde{\Omega}_{\lambda,\beta}$ denote the gridded version of Ω_λ, where entries in the matrix are multiples of $\lambda\beta$.

For any $P \in \Omega_\lambda$, let $\widetilde{P} \in \widetilde{\Omega}_{\lambda,\beta}$ be the matrix obtained by rounding down every element in P.

By the Data Processing Inequality, for any $W \in \mathbb{R}^{d \times k}$, we have

$$d_{TV}\left(p_{W,P}(y|x), p_{W,\widetilde{P}}(y|x)\right) \leq d_{TV}(\mathcal{N}(Wx; P), \mathcal{N}(Wx; \widetilde{P})).$$

By Lemma C.18, we can upper bound the RHS of the above inequality by

$$d_{TV}(\mathcal{N}(Wx; P), \mathcal{N}(Wx; \widetilde{P})) \leq O(d^2 \beta \kappa).$$

Setting

$$\beta = O\left(\frac{\varepsilon}{d^2 \kappa}\right),$$

we have a partition of size $O\left((d^2\kappa/\varepsilon)^{d^2}\right)$ per λ such that:

$$d_{TV}\left(p_{W,P}(y|x), p_{W,\widetilde{P}}(y|x)\right) \leq O(\varepsilon).$$

We will now construct a net over W, so as to show Eqn (43) for all W, P.

W-net. By repeated triangle inequalities, we have

$$|f_n(W, P) - f'_n(W, P)| \leq \left|f_n(W, P) - f_n(W, \widetilde{P})\right| + \left|f_n(W, \widetilde{P}) - f'_n(W, \widetilde{P})\right| + \left|f'_n(W, \widetilde{P}) - f'_n(W, P)\right|.$$

Using the cover over P, the first and last term on the RHS are $O(\varepsilon)$. This gives

$$|f_n(W, P) - f'_n(W, P)| \leq O(\varepsilon) + \left|f_n(W, \widetilde{P}) - f'_n(W, \widetilde{P})\right|. \tag{44}$$

For a fixed $\widetilde{W}, \widetilde{P}$, we will have (43) using a Chernoff bound. Since $\widetilde{P}$ is already finite, we will now construct a net over W *for each* $\widetilde{P}$.

It is sufficient to bound $d_{TV}(p_{W,\widetilde{P}}(y|x_i), p_{W',\widetilde{P}}(y|x_i))$ as the triangle inequality implies that this is larger than the RHS above.

We want, for any W, W' in a cell,

$$|d_{TV}(p_{W,\widetilde{P}}(y|x_i), p_{W^*,P^*}(y|x_i)) - d_{TV}(p_{W',\widetilde{P}}(y|x_i), p_{W^*,P^*}(y|x_i)| \leq O(\varepsilon).$$

for all $i \in [2n]$. It is also sufficient to bound $d_{TV}(p_{W,\widetilde{P}}(y|x_i), p_{W',\widetilde{P}}(y|x_i))$ as the triangle inequality implies that this is larger than the left hand side above.

Lemma C.19 implies that we can find $\mathcal{O}\left(\frac{d}{\epsilon^{3/2}}\sqrt{\log(\frac{2d}{\epsilon})}\right)$ per row of W such that for any W, W' in a cell, either

$$d_{TV}(p_{W,\widetilde{P}}(y|x_i), p_{W^*,P^*}(y|x_i)) \geq d_{TV}(p_{W,\widetilde{P}}(y_j|x_i), p_{W^*,P^*}(y_j|x_i) \geq 1 - \varepsilon$$

which implies

$$d_{TV}(p_{W,\widetilde{P}}(y|x_i), p_{W',\widetilde{P}}(y|x_i) \leq \epsilon$$

or

$$d_{TV}(p_{W,\widetilde{P}}(y|x_i), p_{W',\widetilde{P}}(y|x_i) \leq \sum_j d_{TV}(p_{W,\widetilde{P}}(y|x_i)_j|z_i, p_{W',\widetilde{P}}(y_j|x_i)) \leq \epsilon/d.$$

Therefore, for each i regardless of the value of $W_j^* z_i$ there are at most $\mathcal{O}\left(\frac{d}{\epsilon^{3/2}}\sqrt{\log(\frac{2d}{\epsilon})}\right)$ partitioning hyperplanes.

We then take the intersection of all $2n$ partitions (for each data point z_i). The cells of this partition are defined by $2n$ sets of $\mathcal{O}\left(\frac{d}{\epsilon^{3/2}}\sqrt{\log(\frac{2d}{\epsilon})}\right)$ parallel hyperplanes. Since $z \in \mathbb{R}^k$, the number of cells is at most $\mathcal{O}\left(\left(\frac{nd}{\epsilon^{3/2}}\sqrt{\log(\frac{2d}{\epsilon})}\right)^k\right)$.

Hence the total number of cells for d rows is at most $\mathcal{O}\left(\left(\frac{nd}{\epsilon^{3/2}}\sqrt{\log(\frac{2d}{\epsilon})}\right)^{dk}\right)$.

Putting everything together. Finally, for any $W \in \mathbb{R}^d$ let $\widetilde{W} \in N_{\widetilde{P}}$ be the representative of its cell. Recall that each representative $\widetilde{P}$ of P induces a different cover $N_{\widetilde{P}}$ over W. Let N be the net over the precision matrices P.

By definition of the cells,

$$\left|d_{TV}(p_{W,\widetilde{P}}(y|x_i), p_{W^*,P^*}(y|x_i)) - d_{TV}(p_{\widetilde{W},\widetilde{P}}(y|x_i), p_{W^*,\Sigma^*}(y|x_i))\right| < O(\varepsilon).$$

for all $i \in [2n]$. Thus

$$\left|\left(f'_n(W,\widetilde{P}) - f_n(W,P)\right) - \left(f'_n(\widetilde{W},\widetilde{P}) - f_n(\widetilde{W},P)\right)\right| \leq O(\varepsilon).$$

and so

$$\begin{aligned}
&\Pr[\sup_{W \in \mathbb{R}^{d\times k}, P \in \mathbb{R}^{d\times d}} |f'_n(W,P) - f_n(W,P)| > \varepsilon] \\
&\leq \Pr\left[\max_{w \in N_P, P \in N} |f'_n(W,P) - f_n(W,P)| > \frac{\varepsilon}{4}\right] \\
&\leq e^{\log|N| + \log|N_P| - (\frac{\varepsilon}{4})^2 n/2}
\end{aligned}$$

As there are $\log \kappa \frac{U}{L}$ partitions over P (corresponding to the maximum possible eigenvalue of P), each with $(O(\frac{d^2\kappa}{\varepsilon}))^{d^2}$ elements, we have

$$\log|N| \lesssim d^2 \log\left(\frac{d^2\kappa}{\varepsilon}\right) + \log\log\left(\kappa\frac{U}{L}\right).$$

and each cover N_P over W has size

$$\log|N_P| = 2kd\log\left(\frac{d}{\epsilon^{3/2}}\sqrt{\log\left(\frac{2d}{\epsilon}\right)}\right).$$

This implies that

$$n = C\cdot\left(\frac{kd+d^2}{\varepsilon^2}\right)\log\left(\frac{kd\kappa}{\varepsilon}\log\left(\frac{U}{L}\right)\right),$$

suffices for

$$\Pr[\sup_{W,P} f_n'(W,P) - f_n(W,P) > \varepsilon] < e^{-\Omega(\varepsilon^2 n)}.$$

□

Lemma C.19. *Let $y = \phi(\mu^* + \eta_{\sigma^*})$ where μ^*, σ^* are fixed, and $y_{\mu,\sigma} = \phi(\mu + \eta_\sigma)$. We partition the space $\mathbb{R}$ of μ s.t. for μ, μ' in a cell, either*

$$d_{TV}(p_{\mu,\sigma}(y), p_{\mu',\sigma}(y)) \le \epsilon/2d.$$

or

$$d_{TV}(p_{\mu,\sigma}(y), p_{\mu^*,\sigma^*}(y)) \ge 1-\epsilon \quad \textit{and} \quad d_{TV}(p_{\mu',\sigma}(y), p_{\mu^*,\sigma^*}(y)) \ge 1-\epsilon.$$

Then the number of cells is at most $\mathcal{O}(\frac{d}{\epsilon^{3/2}}\sqrt{\log(\frac{2d}{\epsilon})})$.

Proof. In one dimension

$$d_{TV}(p_{\mu,\sigma}(y), p_{\mu^*,\sigma^*}(y)) = d_{TV}(p_{c\mu,c\sigma}(y), p_{c\mu^*,c\sigma^*}(y))$$

where c is a constant. So, we can assume WLOG that $\sigma^* = 1$. The number of cells in the grid only depends on σ/σ^*.

Now, we show that, regardless of the value of μ^* we only need to make a grid on a segment of length at most $3\max(\sigma,1)\sqrt{\log(2d/\epsilon)}$. This is because for any μ outside the ranges specified below the $d_{TV}(p_{\mu,\sigma}(y), p_{\mu^*,\sigma^*}(y)) \ge 1-\epsilon$.

- If $\mu^* \le -\sqrt{\log(2d/\epsilon)}$ and for any μ such that $\mu \ge \sigma\sqrt{\log(2d/\epsilon)}$, the $d_{TV}(p_{\mu^*,\sigma^*}(y), p_{\mu,\sigma}(y)) \ge$ the difference in the probabilities at 0 which is bigger than $1-\epsilon$.
- If $0 \ge \mu^* \ge -\sqrt{\log(2d/\epsilon)}$ and for any μ s.t. $\mu \ge \max(\sigma,1)\sqrt{\log(2d/\epsilon)}$, the $d_{TV}(p_{\mu^*,\sigma^*}(y), p_{\mu,\sigma}(y))$ is the same as in the linear case and since, $\mu - \mu^* \ge \max(\sigma,1)\sqrt{\log(2d/\epsilon)}$, the $d_{TV}(p_{\mu^*,\sigma^*}(y), p_{\mu,\sigma}(y)) \ge 1-\epsilon$.
- If $0 \le \mu^* \le \sqrt{\log(2d/\epsilon)}$, for any μ s.t. $\mu - \mu^* \ge \max(\sigma,1)\sqrt{\log(2d/\epsilon)}$, the $d_{TV}(p_{\mu^*,\sigma^*}(y), p_{\mu,\sigma}(y)) \ge 1-\epsilon$.
- If $\mu^* \ge \sqrt{\log(2d/\epsilon)}$ then for any μ s.t. $\mu - \mu^* \ge \max(\sigma,1)\sqrt{\log(2d/\epsilon)}$ using the same argument as above, we have, the $d_{TV}(p_{\mu^*,\sigma^*}(y), p_{\mu,\sigma}(y)) \ge 1-\epsilon$. Moreover, this is also true for μ s.t. $-\sigma\sqrt{\log(2d/\epsilon)} \le \mu \le \mu^* - \max(\sigma,1)\sqrt{\log(2d/\epsilon)}$. Therefore, in this case, we have an additional cell.

In addition to the above, for any $\mu, \mu' \le -\sigma\sqrt{\log(2d/\epsilon)}$, the $d_{TV}(p_{\mu,\sigma}(y), p_{\mu',\sigma}(y)) \le \epsilon/2d$ since both $y_{\mu,\sigma}, y_{\mu',\sigma}$ are only non-zero with probability at most $\epsilon/2d$. Therefore, for all the above cases, we only need to partition a segment of length at most $3\max(\sigma,1)\sqrt{\log(2d/\epsilon)}$

Moreover, for σ sufficiently small we can do better. We only need to partition a space of $\sigma\sqrt{\log(2d/\epsilon)}$. This is primarily because when σ sufficiently small, for any μ in the linear case we have that $d_{TV}(p_{\mu,\sigma}(y), p_{\mu^*,\sigma^*}(y)) \ge 1-\epsilon$.

It is easy to see that $d_{TV}(p_{\mu,\sigma}(y), p_{0,1}(y)) \geq d_{TV}(p_{0,\sigma}(y), p_{0,1}(y))$. The PDFs of $\mathcal{N}(0,\sigma)$ and $\mathcal{N}(0,1)$ intersect at $x = \pm\sigma\sqrt{\frac{\log(1/\sigma^2)}{1-\sigma^2}}$. To show that $d_{TV}(p_{0,\sigma}(y), p_{0,1}(y)) \geq 1-\epsilon$, it is now sufficient to show that

$$1 - 2\Phi(-|x|/\sigma) \geq 1-\epsilon \quad \text{and} \quad 1 - 2\phi(-|x|) \leq \epsilon,$$

where $\Phi(x)$ is the CDF of the standard normal distribution. By using classical bounds on $\Phi(x)$, we have that

$$\Phi(-|x|/\sigma) \leq \frac{\exp^{-x^2/2\sigma^2}}{|x|/\sigma} = \frac{\exp^{-\frac{\log(1/\sigma^2)}{2(1-\sigma^2)}}}{|x|/\sigma}$$

which is $\leq \epsilon/2$ if $\sigma^2 \leq \epsilon/2$. And,

$$\Phi(-|x|) \geq \frac{|x|\exp^{-x^2/2}}{x^2+1}$$

which is $\geq (1-\epsilon)/2$ if

When $\mu^* \leq -\sqrt{\log(2d/\epsilon)}$ the same argument as above shows that if $\mu > \sigma\sqrt{\log(2d/\epsilon)}$ then the $d_{TV}(p_{\mu^*,\sigma^*}(y), p_{\mu,\sigma}(y)) \geq$ the difference in the probabilities at 0 which is bigger than $1-\epsilon$.We consider the case where $\mu^* \geq -\sqrt{\log(2d/\epsilon}$.

Since, when σ is small, for any μ in the linear case the TV distance is large it is sufficient to have μ large enough so that the intersection of the PDFs are positive and we are in the linear case.

The point of intersection assuming $\sigma^* = 1$ is given by

$$x = \frac{\mu_1 - \mu_2/\sigma^2 \pm \sqrt{(\mu_1-\mu_2)^2/\sigma^2 + \left(\frac{1}{\sigma^2}-1\right)\log(\sigma^2)}}{\frac{1}{\sigma^2}-1}$$

which is positive whenever

$$\mu_2 \geq \sigma\sqrt{\log(1/\sigma^2 + \mu_1^2} \geq \sigma\sqrt{\log(2d/\epsilon}.$$

For the rest of the space, we partition $\mu \in \mathbb{R}^k$ s.t. for any μ, μ' in a cell,

$$|\mu - \mu'| \leq \sigma\epsilon/2d.$$

This implies that for any μ, μ' in a cell, either,

$$d_{TV}(p_{\mu,\sigma}(y), p_{\mu',\sigma}(y)) \leq \epsilon/2d$$

or

$$d_{TV}(p_{\mu,\sigma}(y), p_{\mu^*,\sigma^*}(y)) \geq 1-\epsilon \text{ and } d_{TV}(p_{\mu',\sigma}(y), p_{\mu^*,\sigma^*}(y)) \geq 1-\epsilon.$$

Then the number of cells is the max of $\mathcal{O}(d\sqrt{\log(2d/\epsilon)}/\epsilon)$ (when σ small) or $\mathcal{O}(d\sqrt{\log(2d/\epsilon)}/\sigma\epsilon) \leq d\sqrt{\log(2d/\epsilon)}/\epsilon^{3/2}$ (when σ large). ☐

D Proof of composition of layers

Proof. We can use the triangle inequality to compose our single layer guarantees. Suppose, for layer j and $j+1$ we have

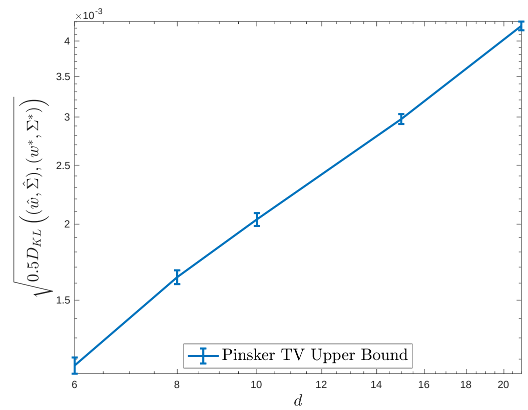
$$d_{TV}(X^j, \hat{X}^j) \leq \epsilon/2 \quad \text{and}$$
$$d_{TV}(\phi(\hat{W}^j_{MLE}X_j + \eta_j), \phi(W^jX_j + \eta_j)) \leq \epsilon/2$$

then,

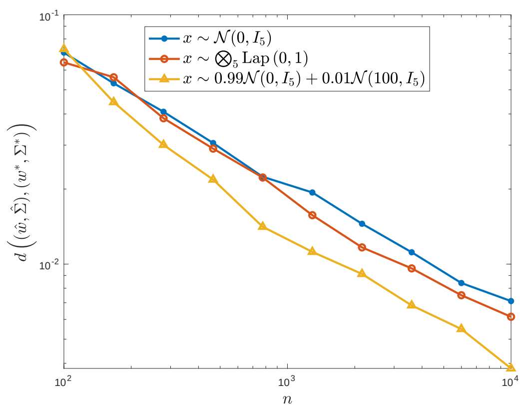
$$\begin{aligned}
&d_{TV}(\phi(\hat{W}^j_{MLE}\hat{X}_j + \eta_j), \phi(W^jX_j + \eta_j)) \\
&\qquad \leq d_{TV}(\phi(\hat{W}^j_{MLE}\hat{X}_j + \eta_j), \phi(\hat{W}^j_{MLE}X_j + \eta_j)) \\
&\qquad + d_{TV}(\phi(\hat{W}^j_{MLE}X_j + \eta_j), \phi(W^jX_j + \eta_j)) \\
&\qquad \leq \epsilon
\end{aligned}$$

where we use the fact that $d_{TV}(f(X), f(Y)) \leq d_{TV}(X, Y)$.

☐



(a) Total Variation Upper Bound vs d



(b) Total Variation vs. n for Different $\mathcal{D}_x$

Figure 4: (a) Plots Pinsker's upper bound on the TV distance as d gets large. We set $\Sigma^* = I_d$ and $W^* = \mathbb{1}_{d\times 1}$, thus setting input dimension $k = 1$. $n = 5000$ samples are taken. As we might expect, the upper bound is increasing in d. each point is determined by 2000 samples. (b) Plot of TV vs. n for additional distributions of x. All three distributions follow roughly the same trend, each point is determined by 2500 samples.

E Simulations

E.1 Additional Simulations

In this section, we provide additional simulations to supplement some of the discussion in Section 5.

E.2 Simulation Details

E.2.1 Figure 2

In these experiment, we set $d = 1$, and plot the results for various values of the number of samples n in Figure 2a and various values of the input dimension k in Figure 2b. For each plot, we fix the true $\sigma^* = 1$ and the $w^* = \mathbb{1}_{k\times 1}$. In each case the MLE is solved via gradient descent with backtracking line search, and we check a first order condition $\|\nabla_{w,\sigma} \log p_{w,\sigma}((y \mid x))\|^2 < \delta = 10^{-3}$ as the exit condition. We verify that increasing or decreasing δ by one order of magnitude makes no difference to the Figure.

The expected total variation distance for the two distributions is calculated as follows. We sample x according to the true distribution (in this case either Laplace or Normal). Then we compute $d_{TV}(p_{\hat{w},\hat{\sigma}}(y \mid x), p_{w,\sigma}(y \mid x))$ via the MATLAB integral function which uses vectorized adaptive quadrature. We repeat this a total of 100 times and take the average to compute our expected total variation. We then repeat the entire process 2000 times, each time optimizing to find an MLE, and then compute its average total variation distance. Lines indicate the average of these experiments, and the error bars, (not easily visible due to their size) indicates one standard error.

E.2.2 Figure 3

In these experiments we fix $d = 3$ to retain reasonable complexity for computing the TV distance, and take input dimension $k = 1$ with deterministic x in order to compare with [35]. In Figure 3a we fix $\Sigma^* = I_d$ and take let $W^* = b\mathbb{1}_{d\times 1}$, where b will vary across our experiments. We set $n = 10000$. In Figure 3b we set $n = 5000$ and adjust Σ such that one diagonal entry is $\kappa^{1/2}$, and the other is $\kappa^{-1/2}$, making the total condition number κ.

In both of these experiments, we restrict the MLE computation to be over diagonal Σ only. This is not because computation of the MLE is too difficult, but rather because computing the TV distance is greatly simplified in this case. The algorithm of Wu et al. is hence modified to use the knowledge that the output must be diagonal. This is simply done, because the procedure of Wu et al. essentially first estimates the diagonal entries of the matrix as if it were diagonal and then computes the correlations. Removing this second phase allows us to achieve our goal.

Since x is deterministic, we do not need to consider randomness in computing the expected TV distance, though other challenges remain. Since our distribution is degenerate, we must be very careful in computing the TV distance in higher dimensions. Specifically, in the diagonal case, the TV may be written as:

$$d_{TV}\left((\widehat{W},\widehat{\Sigma}),(W^*,\Sigma^*)\right) = \frac{1}{2}\int_{\mathbb{R}^d_{\geq 0}} \left|p_{\widehat{W},\widehat{\Sigma}}(y \mid x) - p_{W^*,\Sigma^*}(y \mid x)\right| dy$$
$$= \frac{1}{2}\int_{\mathbb{R}^3_{\geq 0}} \left|\prod_{i=1}^{d} p_{\widehat{W},\widehat{\Sigma}}(y_i \mid x) - \prod_{i=1}^{d} p_{W^*,\Sigma^*}(y_i \mid x)\right| dy,$$

where y_i is the i^{th} element of y. Though at first glace it seems that this is a single high-dimensional integral, the reality is that due to the truncation, the probability mass on the boundary of the non-negative orthant cone $\mathbb{R}^3_{\geq 0}$ has a complex structure that cannot be ignored. Instead we perform a series of integrals of continuous bounded functions, which are much more amenable to Monte-Carlo integration techniques:

$$d_{TV}\left((\widehat{W},\widehat{\Sigma}),(W^*,\Sigma^*)\right) =$$
$$\frac{1}{2}\sum_{S'\in 2^{[d]}} \int_{\mathbb{R}^{|S'|}_{\geq 0}} \left|\prod_{i\in S'} p_{\widehat{W},\widehat{\Sigma}}(y_i \mid x) \prod_{i\in (S')^c} \Phi\left(\widehat{\Sigma}_{ii}^{-1/2}\widehat{W}_i\right) - \prod_{i\in S'} p_{W^*,\Sigma^*}(y_i \mid x) \prod_{i\in (S')^c} \Phi\left(\Sigma_{ii}^{-1/2}W_i\right)\right| dy_{S'}. \tag{45}$$

Essentially, for each possible support of y, S', we integrate over the absolute deviation in those coordinates.

E.2.3 Figure 4

In Figure 4a, we plot an upper bounds for the TV distance of the MLE as the output dimension d grows. We set the input dimension $k = 1$ with deterministic x and fix the number of samples $n = 5000$. To estimate the KL divergence, we repeatedly sample y according to the true distribution, and compute the empirical average log-likelihood ratio.

In Figure 4b we fix the output dimension $d = 1$ and input dimension $k = 5$, and compute the TV over a range of values of n. In addition to x sampled i.i.d. from the Normal and Laplace distributions, we also plot a performance with a Normal mixture, where with probability 0.01, the normal distribution has mean shifted by 100. We observe, as our theory suggests, that in all cases, there is only very minor differences in the expected TV distance.

Note that in the case where x is distributed according to a Normal mixture, we observe that the optimization may become very challenging, and in the plot above, we have omitted some of the instances where optimization failed due to lack of smoothness in the objective and numerical imprecision. Omitting these point may lead to a small systematic error in the figure, which may explain why it is lower than the other plots. In practice, for a fixed optimization budget, we may observe meaningful differences in TV for different distributions of x, since computing the MLE becomes more challenging for more complex heavy-tailed distributions.

F The Likelihood Function

In this section, we discuss the likelihood function, proving log-concavity, as well as discussing computational challenges.

F.1 One Dimensional Case

In this section, we consider the case where the output dimension $d = 1$, with some σ^* and some $W^* \in \mathbb{R}^{1,k}$ and describe how to compute the likelihood function. We defer the proof of the log-concavity to the follwing section, which covers the more general case with $d \geq 1$ When we re-parameterize as $u = -W/\sigma$, $v = 1/\sigma$, the likelihood function is written as:

$$f_{u,v}(y) = -\frac{1}{2}\sum_{i \in S'}(vy_i - u \cdot x_i)^2 + |S'|\log(v) + \sum_{i \in S}\log\Phi(-u \cdot x_i) \tag{46}$$

where in this case we let $S = \{i \mid y_i = 0\}$, where y_i is the i^{th} sample in the set $\{y_i, x_i\}_{i=1}^n$. Note this is distinct from how we define S and S' in the multidimensional case, where it corresponds to the zero and non-zero coordinates of a single sample y_i. The case of $d > 0$ with uncorrelated η follows a similar approach.

Numerical concerns. In (46), the term $\log\Phi(-u \cdot x_i)$ presents some numerical concerns when $u \cdot x_i \gg 0$ if we naively compute $\Phi(-u \cdot x_i)$ and then take the logarithm. Instead we compute it from the *mills ratio* $m(x)$ [27], defined to be the ratio of the standard normal pdf and the complementary cdf. The mills ratio is easily computed, with many well-known expansions, see for example, [13]. Then we can write:

$$\log\Phi(-x) = -\log m(x) - \frac{1}{2}\log(2\pi) - \frac{1}{2}x^2, \quad x > 0.$$

Since $m(x)$ changes relatively slowly in x compared to $\Phi(-x)$, this greatly improves numerical stability.

F.2 Multidimensional Case

In the multi-dimensional case, we will generally use the more standard natural parameters:

$$U := \frac{\Sigma^{-1}}{2},$$
$$v := -\Sigma^{-1}Wx.$$

Note that in the one-dimensional case, we could have also use the natural parameters, but due to the truncation structure, the parameters we used make the computation simpler, in a way that does not apply to the multidimensional case. Also note that here we are considering a fixed x and writing v as a vector. In full generality, we should take $V = -\Sigma^{-1}W$, however, this is a simple extension which we omit here for readability. It turns out that density is log-concave in these natural parameters:

Lemma F.1. *The log-likelihood function in Eqn* (8) *is concave in the natural parameter space.*

Proof. First, let's write the un-truncated density in terms of these parameters:

$$\begin{aligned} f_{W,\Sigma}(y|x) &= \exp\left(-\frac{1}{2}(y - Wx)^T\Sigma^{-1}(y - Wx) - \frac{1}{2}\log|2\pi\Sigma|\right), && (47) \\ &= \exp\left(-\frac{1}{2}y^T\Sigma^{-1}y + x^TW^T\Sigma^{-1}y - x^TW^T\Sigma^{-1}Wx - \frac{1}{2}\log|2\pi\Sigma|\right) && (48) \\ &= \exp\left(-\frac{1}{2}y^TUy - v^Ty - v^TU^{-1}v - \frac{1}{2}\log((2\pi)^n/|U|)\right) && (49) \\ & && (50) \end{aligned}$$

Thus, the untrucated conditional density can be written as:

$$f_{U,v}(y) = \exp\left(-\frac{1}{2}y^TUy + y^Tv - A(U,v)\right),$$

where $A(U,v)$ is the cumulant function (note this is distinct from the related cumulant generating function). A well known result is that A is jointly convex in its arguments, U and v. Taking logs and using this fact, shows us that $f_{U,v}(y)$ is log-concave in U, v.

Our truncated density is simply:

$$f_{U,v}(y|x) = \int_{y_S \leq 0} p_{U,v}(y|x)dy_S,$$

For any log-concave density $f(x)$, integration over a convex subset of the coordinates preserves log-concavity ([9], Example 3.42-3.44). Thus the objective is log-concave. $\square$

Then the likelihood function at U, v can be rewritten as

$$
\begin{aligned}
f_{U,v}(y) &= \log \int_{t_S \leq 0, t_{S'} = y_{S'}} \exp\left(-t^T U t - t^T v - \frac{v^T U^{-1} v}{4} + \frac{1}{2}\log|2U|\right), \\
&= -\frac{v^T U^{-1} v}{4} + \frac{1}{2}\log|2U| + \log \int_{t_S \leq 0, t_{S'} = y_{S'}} \exp\left(-t^T U t - t^T v\right),
\end{aligned}
$$

Separating the terms corresponding to S and S', we get

$$
f_{U,v}(y) = -\frac{v^T U^{-1} v}{4} + \frac{1}{2}\log|2U| - y_{S'}^T U_{S'} y_{S'} - y_{S'}^T v_{S'} + \log \int_{t_S \leq 0} \exp\left(-t_S^T U_S t_S - 2 y_{S'}^T U_{S'S} t_S - v_S^T t_S\right).
$$

The last term resembles the $\log \Phi$ term that appears in the univariate case. This resemblance can be made more clear as follows. Let $r_S = U_{SS'} y_S' + \frac{1}{2} v_S$ and $M^T M = U_S$.

$$
\begin{aligned}
&= \log \int_{t_S \leq 0} \exp\left(-\left(t_S^T M^T M t_S + 2 t_S^T r_S\right)\right) \\
&= \log \int_{t_S \leq 0} \exp\left(-\left((M t_S)^T M t_S + 2 t_S^T r_S + r_S^T M^{-1} M^{-T} r_S - r_S^T M^{-1} M^{-T} r_S\right)\right) \\
&= \log \int_{t_S \leq 0} \exp\left(-\left\|M t_S + M^{-T} r_S\right\|^2 + r_S^T M^{-1} M^{-T} r_S\right) \\
&= r_S^T M^{-1} M^{-T} r_S + \log \int_{t_S \leq 0} \exp\left(-\left\|t_S + U_S^{-1} r_S\right\|_{U_S}^2\right) \\
&= r_S^T U_s^{-1} r_S + \log \int_{t_S \leq 0} \frac{(2\pi)^{d/2} \left|U_S^{-1}/2\right|^{1/2}}{(2\pi)^{d/2} \left|U_S^{-1}/2\right|^{1/2}} \exp\left(-\left\|t_S + U_S^{-1} r_S\right\|_{U_S}^2\right) \\
&= r_S^T U_s^{-1} r_S - \frac{1}{2}\log|2U_S| + \log \int_{t_S \leq 0} \frac{1}{(2\pi)^{d/2} \left|U_S^{-1}/2\right|^{1/2}} \exp\left(-\frac{1}{2}\left\|t_S + U_S^{-1} r_S\right\|_{2U_S}^2\right) + c \\
&= r_S^T U_s^{-1} r_S - \frac{1}{2}\log|2U_S| + \log \Phi\left(0; \mu = -U_S^{-1} r_S, \Sigma = \frac{1}{2} U_S^{-1}\right) + c
\end{aligned}
$$

Putting this together, $f_{U,v}$ can be written as:

$$
\begin{aligned}
f_{U,v}(y) = -\frac{v^T U^{-1} v}{4} + \frac{1}{2}\log|U| - y_{S'}^T U_{S'} y_{S'} - y_{S'}^T v_{S'} + r_S^T U_S^{-1} r_S - \frac{1}{2}\log|U_S| + |S'|\frac{\log(2)}{2} \\
+ \log \Phi\left(0; \mu = -U_S^{-1} r_S, \Sigma = \frac{1}{2} U_S^{-1}\right) + c \quad (51)
\end{aligned}
$$

Thus, it appears that evaluating the likelihood for even a single sample involves the high-dimensional integral that is the rectangular cdf in equation (51).

F.3 Computing Gradients

F.3.1 One Dimensional Case

In the one-dimensional case, the gradient with respect to u is easily computed as:

$$
\nabla_u f_{u,v}(y) = \sum_{i \in S'} (v y_i - u \cdot x_i) x_i - \sum_{i \in S} \frac{1}{m(u \cdot x_i)} x_i,
$$

where we have previously defined $m(x)$ as the mills ratio. Furthermore, we have:

$$
\nabla_v f_{u,v}(y) = |S'|\frac{1}{v} - \sum_{i \in S'} y_i (v y_i - u \cdot x_i)
$$

F.3.2 Multidimensional Case

First we consider the non-integral terms in the likelihood. Differentiating each term wrt U, we get

$$
\begin{aligned}
-\nabla_U \frac{v^T U^{-1} v}{4} &= \frac{1}{4}(U^{-1} v v^T U^{-1}), \\
\nabla_U \frac{1}{2} \log|2U| &= \frac{1}{2} U^{-1}, \\
-\nabla_U y_{S'}^T U_{S'} y_{S'} &= \begin{pmatrix} 0 & 0 \\ 0 & -y_{S'} y_{S'}^T \end{pmatrix}
\end{aligned}
$$

Differentiating each term wrt v, we get

$$
\begin{aligned}
-\nabla_v \frac{v^T U^{-1} v}{4} &= -\frac{1}{2} U^{-1} v, \\
-\nabla_v y_{S'}^T v_{S'} &= \begin{pmatrix} 0 \\ -y_{S'} \end{pmatrix}
\end{aligned}
$$

Now consider the integral term. Differentiating wrt U, we get

$$
\begin{aligned}
&\nabla_U \log \int_{t_S \le 0} \exp\left(-t_S^T U_S t_S - 2 y_{S'}^T U_{S'S} t_S - v_S^T t_S\right) \\
&\qquad = \frac{\int_{t_S \le 0} \begin{pmatrix} -t_S t_S^T & -t_S y_{S'}^T \\ -y_{S'} t_S^T & 0 \end{pmatrix} \exp\left(-t_S^T U_S t_S - 2 y_{S'}^T U_{S'S} t_S - v_S^T t_S\right)}{\int_{t_S \le 0} \exp\left(-t_S^T U_S t_S - 2 y_{S'}^T U_{S'S} t_S - v_S^T t_S\right)}
\end{aligned}
$$

Let M be a matrix such that

$$
M^T M = U_S
$$

Then via completion of squares in the exponential term, we get

$$
\nabla_U \log \int_{t_S \le 0} \exp\left(-t_S^T U_S t_S - 2 y_{S'}^T U_{S'S} t_S - v_S^T t_S\right) \tag{52}
$$

$$
= \frac{\int_{t_S \le 0} \begin{pmatrix} -t_S t_S^T & -t_S y_{S'}^T \\ -y_{S'} t_S^T & 0 \end{pmatrix} \exp\left(-\|M t_S + (M^{-1})^T \left(U_{SS'} y_{S'} + \frac{v_S}{2}\right)\|^2\right)}{\int_{t_S \le 0} \exp\left(-\|M t_S + (M^{-1})^T \left(U_{SS'} y_{S'} + \frac{v_S}{2}\right)\|^2\right)} \tag{53}
$$

Notice that this density is Gaussian, with mean and covariance:

$$
\mathcal{N}\left(-M^{-1} \left(M^{-1}\right)^T \left(U_{SS'} y_{S'} + \frac{v_S}{2}\right); \frac{U_S^{-1}}{2}\right).
$$

And hence, the gradient can be estimated as

$$
\nabla_U \log \int_{t_S \le 0} \exp\left(-t_S^T U_S t_S - 2 y_{S'}^T U_{S'S} t_S - v_S^T t_S\right) = \mathop{\mathbb{E}}_{t_S}\left[\begin{pmatrix} -t_S t_S^T & -t_S y_{S'}^T \\ -y_{S'} t_S^T & 0 \end{pmatrix}\right]
$$

where t_S is the truncation of

$$
z \sim \mathcal{N}\left(-M^{-1} \left(M^{-1}\right)^T \left(U_{SS'} y_{S'} + \frac{v_S}{2}\right); \frac{U_S^{-1}}{2}\right).
$$

to the negative quadrant. A similar calculation gives the gradient for v as

$$
\nabla_v \log \int_{t_S \le 0} \exp\left(-t_S^T U_S t_S - 2 y_{S'}^T U_{S'S} t_S - v_S^T t_S\right) = \mathop{\mathbb{E}}_{t_S}\left[\begin{pmatrix} -t_S \\ 0 \end{pmatrix}\right]
$$