

Appendix

Lemma 1 (a) If random vector \mathbf{X} is symmetric about 0, then $A\mathbf{X} + \mu$ is symmetric about μ .
(b) If \mathbf{X}, \mathbf{Y} are independent and both symmetric about 0, $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ is also symmetric about 0.

Proof: (a). Since \mathbf{X} is symmetric about 0, for all y , $\Pr(A\mathbf{X} + \mu \geq \mu + y) = \Pr(A\mathbf{X} \geq y) = \int \mathbf{1}_{Ax \geq y} du_x = \int \mathbf{1}_{-Ax \geq y} du_x = \Pr(A\mathbf{X} \leq -y) = \Pr(A\mathbf{X} + \mu \leq \mu - y)$. From the definition of symmetry, we know $A\mathbf{X} + \mu$ is symmetric about μ .

(b). Since \mathbf{X} and \mathbf{Y} are independent and both symmetric about 0, for all z , $\Pr(\mathbf{Z} \geq z) = \iint \mathbf{1}_{x+y \geq z} du_x du_y = \int du_x \int \mathbf{1}_{y \geq z-x} du_y = \int du_x \int \mathbf{1}_{y \leq x-z} du_y = \int du_y \int \mathbf{1}_{x \geq y+z} du_x = \int du_y \int \mathbf{1}_{x \leq -y-z} du_x = \iint \mathbf{1}_{x+y \leq -z} du_x du_y = \Pr(\mathbf{Z} \leq -z)$. From the definition of symmetry, we know \mathbf{Z} is symmetric about 0. ■

Equation (26) in Proposition 3

$$\max_{\mathbf{x} \sim (\mu, \sigma^2)_{SU}} \mathbb{E}[(\mathbf{x} - t)_+] = \begin{cases} \frac{(\sqrt{3}\sigma - t + \mu)^2}{4\sqrt{3}\sigma}, & \text{if } \mu - \frac{\sigma}{\sqrt{3}} \leq t \leq \mu + \frac{\sigma}{\sqrt{3}} \\ \frac{\sigma^2}{9(t-\mu)}, & \text{if } t > \mu + \frac{\sigma}{\sqrt{3}} \\ -\frac{\sigma^2 + 9(t-\mu)^2}{9(t-\mu)}, & \text{if } t < \mu - \frac{\sigma}{\sqrt{3}} \end{cases}$$

Proof: Without loss of generality, we assume $\mu = 0$, otherwise we can explicitly achieve this by translation $x' = x - \mu, t' = t - \mu$. We use the technique established in [13] to prove (26).

Using the strong duality property between moment problems and linear programming [13], $\max_{\mathbf{x} \sim (0, \sigma^2)_{SU}} \mathbb{E}[(x - t)_+]$ is equivalent to:

$$\min_{y_0, y_1} \quad y_0 + y_1 \sigma^2 \quad (27)$$

$$\text{s.t.} \quad y_0 p + \frac{y_1 p^3}{3} \geq \frac{1}{2} \int_{-p}^p [x - t]_+ dx, \quad \forall p \geq 0 \quad (28)$$

1). Consider $t \geq 0$.

Constraint (28) now can be simplified as:

$$\begin{cases} y_0 p + \frac{y_1 p^3}{3} \geq 0, & 0 \leq p \leq t \\ y_0 p + \frac{y_1 p^3}{3} \geq \frac{(p-t)^2}{4}, & p \geq t \end{cases}$$

The first condition implies $y_0 \geq 0$ and the second implies $y_1 \geq 0$. Therefore problem (27)-(28) can be rewritten as:

$$\min_{y_0 \geq 0, y_1 \geq 0} \quad y_0 + y_1 \sigma^2 \quad (29)$$

$$\text{s.t.} \quad y_0 p + \frac{y_1 p^3}{3} - \frac{(p-t)^2}{4} \geq 0, \quad \forall p \geq t \quad (30)$$

Denote

$$\begin{aligned} h(p) &= \frac{y_1 p^3}{3} - \frac{p^2}{4} + (y_0 + \frac{t}{2})p - \frac{t^2}{4} \\ h'(p) &= y_1 p^2 - \frac{p}{2} + (y_0 + \frac{t}{2}) \end{aligned}$$

Obviously, $h(t) \geq 0, h'(t) \geq 0$. The constraint (30) can be satisfied either

- $y_0 + t/2 - 1/(16y_1) \geq 0$, which means $\forall p, h'(p) \geq 0$, or
- $y_0 + t/2 - 1/(16y_1) \leq 0$ but the minimum of $h(p)$ is no less than 0.

The first case amounts to:

$$\begin{aligned} \min_{y_0 \geq 0, y_1 \geq 0} \quad & y_0 + y_1 \sigma^2 \\ \text{s.t.} \quad & y_0 + t/2 - 1/(16y_1) \geq 0 \end{aligned}$$

and the minimum is (exactly the same as that of symmetric distributions)

$$\begin{cases} (\sigma - t)/2, & 0 \leq t \leq \sigma/2 \\ \frac{\sigma^2}{8t}, & t \geq \sigma/2 \end{cases} \quad (31)$$

For the second case, let $p_+ = \frac{1 + \sqrt{1 - 8ty_1 - 16y_0y_1}}{4y_1}$ be the larger minimizer of $h(p)$, problem (29)-(20) now becomes:

$$\begin{aligned} \min_{y_1 \geq 0} \quad & y_0 + y_1 \sigma^2 \\ \text{s.t.} \quad & 0 \leq y_0 \leq \frac{1}{16y_1} - \frac{t}{2} \\ & h(p_+) \geq 0 \Rightarrow \frac{t^3 - 30t^2y_0 - 96ty_0^2 - 64y_0^3 + (t^2 + 16ty_0 + 16y_0^2)^{3/2}}{18t^4} \leq y_1 \leq \frac{1}{8t} \end{aligned}$$

The minimum is (thanks to Mathematica!)

$$\begin{cases} -t/2 + \frac{3\sigma^2 + t^2}{4\sqrt{3}\sigma}, & 0 \leq t \leq \sigma/\sqrt{3} \\ \frac{\sigma^2}{9t}, & t \geq \sigma/\sqrt{3} \end{cases} \quad (32)$$

Comparing (31) and (32), we know the latter gives the minimum of problem (29)-(30).

2). Consider $t \leq 0$.

Similarly, problem (27)-(28) can be rewritten as:

$$\begin{aligned} \min_{y_0 \geq -t, y_1 \geq 0} \quad & y_0 + y_1 \sigma^2 \\ \text{s.t.} \quad & y_0 p + \frac{y_1 p^3}{3} - \frac{(p - t)^2}{4} \geq 0, \quad \forall p \geq -t \end{aligned}$$

Using similar procedures as when $t \geq 0$, we know the minimum is:

$$\begin{cases} -t/2 + \frac{3\sigma^2 + t^2}{4\sqrt{3}\sigma}, & -\sigma/\sqrt{3} \leq t \leq 0 \\ -t - \frac{\sigma^2}{9t}, & t \leq -\sigma/\sqrt{3} \end{cases}$$

Combining the above two cases finishes the proof. ■