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# Supplementary Material for “Combinatorial Optimization with Graph Convolutional Networks and Guided Tree Search”

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## A Problem reductions

As mentioned in the main text, the MVC, MC, and SAT problems can all be represented as instances of the MIS problem:

**MVC  $\rightarrow$  MIS.** Given a graph, the minimum vertex cover and the maximal independent set are complementary. A vertex set is independent if and only if its complement is a vertex cover, and thus the solutions of MIS and MVC are complementary to each other [6].

**MC  $\rightarrow$  MIS.** The maximal clique of a graph is the maximal independent set of the complementary graph [6].

**SAT  $\rightarrow$  MIS.** Given a SAT instance, we construct a graph as follows. Each literal in a clause is a vertex in the graph. Vertices in the same clause are adjacent to each other in the graph. An edge is spanned between two vertices in different clauses if they represent literals where one is the negation of the other. With this graph, if we can find an independent set with size equal to the number of clauses, the formula is satisfiable; otherwise, it is not. The independent set also specifies the truth assignment to the variables [3].

## B Algorithms

The complete basic algorithm is summarized in Algorithm 1, the revised algorithm with diversity and tree search is summarized in Algorithm 2, and the variant with parallelized tree search is summarized in Algorithm 3.

## C Classic elements

### C.1 Local search

For local search, we adopt a 2-improvement local search algorithm [2, 4]. This algorithm iterates over all vertices in the graph and attempts to replace a 1-labelled vertex  $v_i$  with two 1-labelled vertices  $v_j$  and  $v_k$ . In MIS,  $v_j$  and  $v_k$  must be neighbors of  $v_i$  that are 1-tight and not adjacent to each other. Here, a vertex is 1-tight if exactly one of its neighbors is labelled 1. In other words,  $v_i$  is the only 1-labelled neighbor of  $v_j$  and  $v_k$  in the graph. By using a data structure that allows inserting and deleting nodes in time proportional to their degrees, this local search algorithm can find a valid 2-improvement in  $O(E)$  time if it exists. An even faster incremental version of the algorithm maintains a list of candidate nodes that are involved in 2-improvements. It ensures that a node is not repeatedly examined unless there is some change in its neighborhood.

### C.2 Graph reduction

For graph reduction, some efficient reduction techniques [1, 5] we adopted are given below:

- **Pendant vertices:** A vertex  $v$  of degree one is called a pendant, and it must be in some MIS. Thus,  $v$  and its neighbors can be removed from  $\mathcal{G}$ .
- **Vertex folding:** For a 2-degree vertex  $v$  whose neighbors  $u$  and  $w$  are not adjacent, either  $v$  is in some MIS, or both  $u$  and  $w$  are in some MIS. Thus, we can merge  $u$ ,  $v$  and  $w$  to a single vertex  $v'$  and decide which vertices are in the MIS later.

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**Algorithm 1** BasicMIS

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**Input:** Graph  $\mathcal{G}$ **Output:** Labelling of all the vertices in  $\mathcal{G}$ 

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1:  $v \leftarrow \text{argsort}(f(\mathcal{G}; \theta))$  in a descending order
2: for  $i := 1$  to  $\|v\|$  do
3:   if vertex  $v(i)$  is already labelled in  $\mathcal{G}$  then
4:     break
5:   end if
6:   Label  $v(i)$  as 1 in  $\mathcal{G}$ 
7:   Label the neighbors of  $v(i)$  as 0 in  $\mathcal{G}$ 
8: end for
9:  $\mathcal{G}' \leftarrow$  residual graph of  $\mathcal{G}$  by removing labelled vertices
10: if  $\mathcal{G}'$  is not empty then
11:   Run BasicMIS on  $(\mathcal{G}')$ 
12: end if

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**Algorithm 2** MIS with diversity and tree search

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**Input:** Graph  $\mathcal{G}$ **Output:** The best solution found

```

1: Initialize queue  $\mathcal{Q}$  with  $\mathcal{G}$ 
2: while execute time is under budget do
3:    $\mathcal{G}' \leftarrow \mathcal{Q}.\text{random\_pop}()$ 
4:   for  $m := 1$  to  $M$  do
5:      $v \leftarrow \text{argsort}(f^m(\mathcal{G}'; \theta))$  in a descending order
6:     for  $i := 1$  to  $\|v\|$  do
7:       if vertex  $v(i)$  is already labelled in  $\mathcal{G}'$  then
8:         break
9:       end if
10:      Label  $v(i)$  as 1 in  $\mathcal{G}'$ 
11:      Label the neighbors of  $v(i)$  as 0 in  $\mathcal{G}'$ 
12:    end for
13:    if  $\mathcal{G}'$  is completely labelled then
14:      Update the current best solution
15:    else
16:      Remove labelled vertices from  $\mathcal{G}'$ 
17:      Add  $\mathcal{G}'$  to  $\mathcal{Q}$ 
18:    end if
19:  end for
20: end while

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- Unconfined: Define  $\mathcal{N}(\cdot)$  as the neighbours of a vertex or set. A vertex  $v$  is unconfined when determined by the following rules. First, let set  $\mathcal{S} = \{v\}$ . Then, we find a  $u \in \mathcal{N}(\mathcal{S})$  such that  $|\mathcal{N}(u) \cap \mathcal{S}| = 1$  and  $|\mathcal{N}(u) \setminus \mathcal{S}|$  is minimized. If there is no such vertex, then  $v$  is confined. If  $\mathcal{N}(u) \setminus \mathcal{S} = \Phi$ , then  $v$  is unconfined. If  $\mathcal{N}(u) \setminus \mathcal{S}$  is a single vertex  $w$ , then add  $w$  to  $\mathcal{S}$  and repeat the algorithm. Since there always exists an MIS without no unconfined vertices, they can be removed from  $\mathcal{G}$ .
- Twin: Define  $\mathcal{G}[\mathcal{S}]$  as a subgraph of  $\mathcal{G}$  induced by a set of vertices  $\mathcal{S}$ . Let  $u$  and  $v$  be vertices of degree 3 with  $\mathcal{N}(u) = \mathcal{N}(v)$ . If  $\mathcal{G}[\mathcal{N}(u)]$  has edges,  $u$  and  $v$  must be in the MIS. If  $\mathcal{G}[\mathcal{N}(u)]$  has no edges, some vertices in  $\mathcal{N}(u)$  may belong to some MIS. In this case, we can still remove  $u$ ,  $v$ ,  $\mathcal{N}(u)$  and  $\mathcal{N}(v)$  from  $G$ , and add a new gadget vertex  $w$  to  $\mathcal{G}$  with edges to  $u$ 's order-2 neighbors (vertices at a distance 2 from  $u$ ). Later if  $w$  is in the computed MIS, then none of  $u$ 's order-2 neighbors are in the MIS, and therefore  $\mathcal{N}(u)$  is in the MIS. If  $w$  is not in the computed MIS, then some of  $u$ 's order-2 neighbors are in the MIS, and therefore  $u$  and  $v$  are in the MIS.

## D Real-world graphs

Table 1 provides the statistics and descriptions of the real-world graphs used in our experiments.

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**Algorithm 3** MIS with diversity and parallelized tree search

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**Input:** Graph  $\mathcal{G}$ , thread number  $T$ **Output:** The best solution found

```
1: while execute time is under budget do
2:   if queue  $\mathcal{Q}$  is empty then
3:     Initialize  $\mathcal{Q}$  with  $\mathcal{G}$ 
4:   end if
5:   for all thread  $t := 1$  to  $T$  do
6:      $\mathcal{G}' \leftarrow \mathcal{Q}.\text{random\_pop}()$ 
7:      $m := \text{random\_integer}(M)$ 
8:      $v \leftarrow \text{argsort}(f^m(\mathcal{G}'; \theta))$  in a descending order
9:     for  $i := 1$  to  $\|v\|$  do
10:      if vertex  $v(i)$  is already labelled in  $\mathcal{G}'$  then
11:        break
12:      end if
13:      Label  $v(i)$  as 1 in  $\mathcal{G}'$ 
14:      Label the neighbors of  $v(i)$  as 0 in  $\mathcal{G}'$ 
15:    end for
16:    if  $\mathcal{G}'$  is completely labelled then
17:      Update the current best solution
18:    else
19:      Remove labelled vertices from  $\mathcal{G}'$ 
20:      Add  $\mathcal{G}'$  to  $\mathcal{Q}$ 
21:    end if
22:  end for
23: end while
```

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Name	Nodes	Edges	Description
ego-Facebook	4,039	88,234	Social circles from Facebook (anonymized)
ego-Gplus	107,614	13,673,453	Social circles from Google+
ego-Twitter	81,306	1,768,149	Social circles from Twitter
soc-Epinions1	75,879	508,837	Who-trusts-whom network of Epinions.com
soc-Slashdot0811	77,360	905,468	Slashdot social network from November 2008
soc-Slashdot0922	82,168	948,464	Slashdot social network from February 2009
wiki-Vote	7,115	103,689	Wikipedia who-votes-on-whom network
wiki-RfA	10,835	159,388	Wikipedia Requests for Adminship (with text)
bitcoin-otc	5,881	35,592	Bitcoin OTC web of trust network
bitcoin-alpha	3,783	24,186	Bitcoin Alpha web of trust network
Citeseer	3,327	4,732	Citation network from Citeseer
Cora	2,708	5,429	Citation network from Cora
Pubmed	19,717	44,338	Citation network from PubMed

Table 1: Real-world graph statistics and descriptions.

## E Network width $C$

We analyze the effect of the width  $C$  of the intermediate layers. Table 2 shows the fraction of solved problems and average size of the computed MIS solution on the SATLIB validation set for  $C = 16, 32, 64, 128$ . The performance decreases when  $C = 64, 128$ , probably because a complex model is more computationally expensive and thus there is less time for the search algorithm.  $C = 32$  provides the best balance of performance and efficiency.

## References

- [1] Takuya Akiba and Yoichi Iwata. Branch-and-reduce exponential/FPT algorithms in practice: A case study of vertex cover. In *ALENEX*, 2015.

	$C = 16$	$C = 32$	$C = 64$	$C = 128$
Solved	97.0%	<b>98.8%</b>	97.6%	97.2%
MIS	426.86	<b>426.88</b>	426.87	426.87

Table 2: Effect of the hyperparameter  $C$ .

- [2] Diogo Vieira Andrade, Mauricio G. C. Resende, and Renato Fonseca F. Werneck. Fast local search for the maximum independent set problem. *J. Heuristics*, 18(4), 2012.
- [3] Sanjoy Dasgupta, Christos H. Papadimitriou, and Umesh V. Vazirani. *Algorithms*. McGraw-Hill, 2008.
- [4] Thomas A. Feo, Mauricio G. C. Resende, and Stuart H. Smith. A greedy randomized adaptive search procedure for maximum independent set. *Operations Research*, 42(5), 1994.
- [5] Sebastian Lamm, Peter Sanders, Christian Schulz, Darren Strash, and Renato F. Werneck. Finding near-optimal independent sets at scale. *J. Heuristics*, 23(4), 2017.
- [6] Steven Skiena. *The Algorithm Design Manual*. Springer, 2008.