A Proof of Proposition 1

Note that in problem (7) the term $(a_{ii} - d_i d_i/(2m))v_i^T v_i$ is constant because $\|v_i\| = 1$. Thus, in the subproblem $Q(v_i)$ for variable v_i, we can ignore the constant term and write the gradient $\nabla Q(v_i)$ as

$$\nabla Q(v_i) = \frac{1}{2m} \sum_{j \neq i} \big(a_{ij} - \frac{d_i d_j}{2m} \big) \; v_j. \tag{13}$$

Further, since there is no v_i term in $\nabla Q(v_i)$, the objective function $Q(v_i)$ for the subproblem of variable v_i becomes $q^T v_i$ with $q = \nabla Q(v_i)$, up to a constant. For simplicity, denote v_i as v, and the subproblem reduces to

$$\underset{v}{\text{maximize}} \;\; q^T v, \;\; \text{s.t. } v \in \mathbb{R}_+^r, \; \|v\| = 1, \; \mathrm{card}(v) \leq k. \tag{14}$$

Let v^* be the optimal solution of the above subproblem (8) (existence by compactness). When $q \leq 0$, we have $\max(q) \leq 0$. With $\|v\|_2 = 1$, $v \geq 0$, and $\|v\|_2 \leq \|v\|_1$, there is

$$\max(q) = \max(q)\|v\|_2 \geq \max(q)\|v\|_1 = \max(q) \sum_t v_t \geq \sum_t q_t v_t = q^T v. \tag{15}$$

Thus, $e(t)$ with the max q_t is the optimal solution in the first case. For the second case, there is at least one coordinate p such that $q_p > 0$. Now we exclude the following two cases of inactive coordinates by contradictions.

(When $q_t < 0$) We know $v_t^* = 0$. Otherwise, suppose there is a $v_t^* > 0$ with $q_t < 0$.

If $q^T v^* \leq 0$, selecting $v^* = e(p)$ violates the optimality of v^*, a contradiction.

If $q^T v^* > 0$, we have

$$0 < q^T v^* < q^T (v^* - e(t)v_t^*) \leq q^T (v^* - e(t)v_t^*)/\|v^* - e(t)v_t^*\|, \tag{16}$$

also a contradiction to the optimality of v^*, because the last term is a feasible solution.

(When $q_t < q_{[k]}$, where $q_{[k]}$ is the k-th largest value) We know $v_t^* = 0$. Otherwise, there must be a coordinate j in the top-k-largest value that is not selected ($v_j^* = 0$) because $\mathrm{card}(v^*) \leq k$. This way, we have

$$q^T v^* < q^T (v^* - e(t)v_t^* + e(j)v_t^*), \tag{17}$$

which contradicts to the optimality of v^* because $(v^* - e(t)v_t^* + e(j)v_t^*)$ is a feasible solution.

Thus, by removing the inactive coordinates, the effective objective function $q^T v^*$ becomes $\text{top}_k^+(q)^T v^*$, and the optimal solution follows from $\|v_i^*\| = 1$ and $\text{top}_k^+(q) \geq 0$. $\square$

B Proof of Theorem 2

Define the projected gradient (for maximization) as

$$\texttt{grad}(V) = P_\Omega(V + \nabla Q(V)) - V, \tag{18}$$

where P_Ω is the projection (under 2-norm) to the constraint set Ω of the optimization problem (7)

$$\Omega = \{V \mid v_i \in \mathbb{R}^r_+,\ \|v_i\| = 1,\ \mathrm{card}(v_i) \leq k,\ \forall i = 1, \ldots, n\}, \tag{19}$$

and denote Ω_i as the constraint for v_i for the separable Ω. Because the cardinality constraint is an union between finite hyperplanes, it is a closed set, which implies the constraint of the optimization problem is a compact set. Thus, by the Weierstrass extreme value theorem, the function $Q(V)$ is upper-bounded and must attain global maximum over the constraint.

Now we connect the exact update in the Locale algorithm with the projected gradient. Denote v_i^+ as the update taken for the subproblem $Q(v_i)$. Because the Locale algorithm performs an exact update (Proposition 1), we have

$$\nabla Q(v_i)^T v_i^+ \geq \nabla Q(v_i)^T u,\ \forall u \in \Omega_i. \tag{20}$$

Further, because $\|v_i^+\|^2 = 1$ and $\|u\|^2 = 1$, we have

$$\|v_i^+ - \nabla Q(v_i)\|^2 \leq \|u - \nabla Q(v_i)\|^2,\ \forall u \in \Omega_i. \tag{21}$$

This means that the update v_i^+ is the projection of $\nabla Q(v_i)$ to the constraint set Ω_i. To connect the update with the projected gradient, we need the following lemma.

Lemma 4. *Denote the projection (under 2-norm) of a point x on a closed constraint set Ω as $P_\Omega(x)$. Then for any scalar $\alpha > 1$ and vector q, we have*

$$q^T(P_\Omega(x + \alpha q) - P_\Omega(x + q)) \geq 0$$

The proof is listed in Appendix C. Taking the lemma with $\alpha \to \infty$ and let $q = \nabla Q(v_i)$, we have

$$0 \leq \lim_{\alpha \to 0} q^T(P_{\Omega_i}(v_i + \alpha q) - P_{\Omega_i}(x + q)) = q^T(v_i^+ - P_{\Omega_i}(v_i + q)), \tag{22}$$

where the last equation follows because v_i^+ is the projection of q on Ω_i with $\|\cdot\| = 1$ constraint [4]. Further, apply the definition of projection $P_{\Omega_i}(v_i + q)$ again on the feasible v_i, we have

$$\|P_{\Omega_i}(v_i + q) - (v_i + q)\|^2 \leq \|v_i - (v_i + q)\|^2, \tag{23}$$

and after rearranging there is

$$\|P_{\Omega_i}(v_i + q) - v_i\|^2 \leq 2q^T(P_{\Omega_i}(v_i + q) - v_i). \tag{24}$$

Applying (22) to the equation above, we have

$$\|P_{\Omega_i}(v_i + q) - v_i\|^2 \leq 2q^T(v_i^+ - v_i). \tag{25}$$

The right hand side of the above equation equals the function increment $Q(v_i^+) - Q(v_i)$. Thus,

$$\|P_{\Omega_i}(v_i + q) - v_i\|^2 \leq 2(Q(v_i^+) - Q(v_i)). \tag{26}$$

Now, taking expectation over the random coordinate i, we have

$$\frac{1}{n}\|P_\Omega(V + \nabla Q(V)) - V\|^2 = \mathbb{E}\|P_{\Omega_i}(v_i + q) - v_i\|^2 \leq 2\mathbb{E}(Q(v_i^+) - Q(v_i)) = Q(V^{t+1}) - Q(V^t). \tag{27}$$

Further, since $Q(V^{t+1}) - Q(V^t)$ is monotonic increasing, summing them over iterations 0 to $T-1$ forms a telescoping sum, which is upper-bounded by $Q(V^*) - Q(V^0)$, where V^* is the global optimal solution of $Q(V)$. Substitute the definition of projected gradient (18), we have

$$\frac{T}{n} \min_t \|\texttt{grad}(V^t)\|^2 \leq \frac{1}{n} \sum_{t=0}^{T-1} \|\texttt{grad}(V^t)\|^2 \leq 2(Q(V^*) - Q(V^0)). \tag{28}$$

Thus, the projected gradient $\texttt{grad}(V)$ converges to zero at a $O(1/T)$ rate. $\square$

[4]Note that in Proposition 1, when $q \leq 0$ and there are multiple maximum q_t, we further select the t with the maximum $(v_i)_t$ in the previous iteration. This makes the limit to hold on the corner case $q = 0$.

C Proof for Lemma 4

By definition of the projection $P_\Omega(x+q)$, we have

$$\|P_\Omega(x+q)-(x+q)\|^2 \leq \|P_\Omega(x+\alpha q)-(x+q)\|^2.$$

Take out the q term out of the norm and rearrange, there is

$$\|P_\Omega(x+q)-x\|^2 \leq \|P_\Omega(x+\alpha q)-x\|^2 - 2q^T(P_\Omega(x+\alpha q)-P_\Omega(x+q)). \tag{29}$$

Similarly, by definition of the projection $P_\Omega(x+\alpha q)$, there is

$$\|P_\Omega(x+\alpha q)-x\|^2 \leq \|P_\Omega(x+q)-x\|^2 - 2\alpha q^T(P_\Omega(x+q)-P_\Omega(x+\alpha q)). \tag{30}$$

Sum (29) and (30), the norms cancel, and we have

$$2(\alpha-1)q^T(P_\Omega(x+\alpha q)-P_\Omega(x+q)) \geq 0,$$

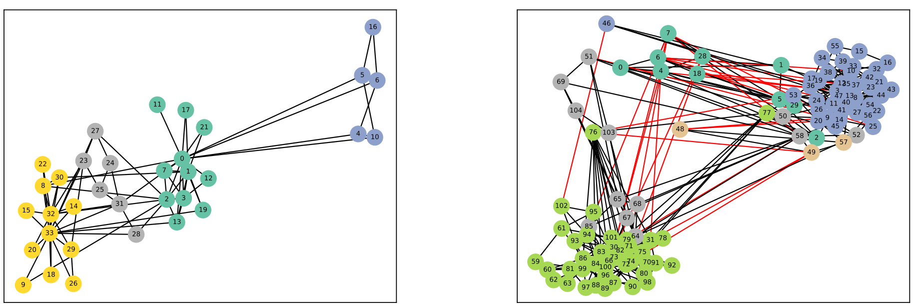
which implies

$$q^T(P_\Omega(x+\alpha q)-P_\Omega(x+q)) \geq 0. \tag{31}$$

Thus, the result holds. □

D Experiments on networks with ground truth

In this section, we compare results from the Leiden-Locale method on data with the ground truth for partitions. The result is listed in Figure 4.



(a) `zachary` (ground truth = 4 clusters) (b) `polbook` (ground truth = 3 clusters)

Figure 4: The comparison of the results from Leiden-Locale method to ground-truth partitions in the `zachary` and `polbook` datasets. The position of each node is arranged using the 2D Fruchterman-Reingold force-directed algorithm from the ground-truth using `networkx` [25], and the color of each node indicates the solution community given by Leiden-Locale algorithm. The red edges between nodes indicates the case when two nodes are inside the same cluster in the ground truth but wasn't assigned so in our algorithm. For `zachary`, the Leiden-Locale algorithm returns a perfect answer comparing to the ground truth with a perfect modularity of 0.4197 [32]. For `polbook`, it misclassifies 18 over 105 nodes, but still attains a best known modularity of 0.5272 [2].

E Pseudo-code for the Leiden-Locale algorithm

Here we list the pseudo-code for the Leiden-Locale method. Note that we reuse Algorithm 2 in Algorithm 3–4 for rounding and refinement by changing its constraint and initialization. And in the actual code, Algorithm 3–4 are combined as a single subroutine.

Algorithm 2 Optimization procedure for the Locale algorithm

1: **procedure** LOCALEEMBEDDINGS(Graph G, Partition P)
2: Initialize V with $v_i = e(i),\ i = 1, \dots, n.$
3: Initialize the ring queue R with indices $i = 1, \dots, n.$
4: Let $z = \sum_{j=1}^{n} d_j v_j.$
5: **while** not yet converged **do**
6: $i = R.pop()$ ▷ Pick an index from the ring queue
7: $\nabla Q(v_i) = \sum_{j \in P(i)} a_{ij} v_j - \frac{d_i}{2m}(z - d_i v_i)$ ▷ Sums only j in the same partition of i
8: $g_i = \begin{cases} e(t) \text{ with the max } (\nabla Q(v_i))_t, & \text{if } \nabla Q(v_i) \leq 0, \\ \text{top}_k^+(\nabla Q(v_i)), & \text{otherwise.} \end{cases}$
9: $v_i^{\text{old}} = v_i,\ \ v_i = g_i / \|g_i\|$ ▷ Perform the closed-form update
10: $z = z + d_i(v_i - v_i^{\text{old}})$ ▷ Maintain the z
11: Push all neighbors j with nonzero a_{ij} into the ring queue R if it is not already inside.
12: **end while**
13: **return** the embedding V
14: **end procedure**

Algorithm 3 Rounding procedure for the Locale algorithm

procedure LOCALEROUNDING(Graph G, Partition P, Embedding E)
Initialize V with input E.
Run line 3–12 of Algorithm 2 with cardinality constraint $k = 1$.
Let the index of the 1-sparse embedding above be the new partition P'.
return P'
end procedure

Algorithm 4 Refine and Aggregate procedure from the Leiden algorithm

procedure LEIDENREFINEAGGREGATE(Graph G, Partition P)
Refine $P' \leftarrow \textit{LocaleRounding}(G, P)$ by restricting the local move within its partition.[5]
Forms a hypergraph G' by merging nodes inside the same partitions in P' and simplify P'.
done $\leftarrow |P|$ equals $|G'|$.
return G', P', done
end procedure

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[5]This is the refinement step implemented in the package `python-leiden`.