Supplementary Material of "Bayesian Causal Structural Learning with Zero-Inflated Poisson Bayesian Networks"

Junsouk Choi
Department of Statistics
Texas A&M University
College Station, TX 77843
jchoi@stat.tamu.edu

Robert Chapkin
Department of Nutrition
Texas A&M University
College Station, TX 77843
r-chapkin@tamu.edu

Yang Ni
Department of Statistics
Texas A&M University
College Station, TX 77843
yni@stat.tamu.edu

A. Proof of Theorem 1

We provide a detailed proof for Theorem 1. Consider two Markov equivalent ZIPBNs $\mathcal{B} = (\mathcal{G}, \boldsymbol{\theta})$ and $\mathcal{B}' = (\mathcal{G}', \boldsymbol{\theta}')$ with $\mathcal{G} = (V, \boldsymbol{E}), \mathcal{G}' = (V, \boldsymbol{E}' \neq \boldsymbol{E}), \boldsymbol{\theta} = \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\gamma}\}$, and $\boldsymbol{\theta}' = \{\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}', \boldsymbol{\gamma}'\}$. Let $p\left(\boldsymbol{X}|\boldsymbol{E}, \boldsymbol{\theta}\right)$ and $p\left(\boldsymbol{X}|\boldsymbol{E}', \boldsymbol{\theta}'\right)$ denote the sampling distribution of $\boldsymbol{X}$ under $\mathcal{B}$ and $\mathcal{B}'$, respectively. We will prove by contradiction that $\mathcal{B}$ and $\mathcal{B}'$ are not distribution equivalent,

$$p\left(\boldsymbol{X}|\boldsymbol{E}, \boldsymbol{\theta}\right) \neq p\left(\boldsymbol{X}|\boldsymbol{E}', \boldsymbol{\theta}'\right).$$

Suppose, on the contrary, that we can find a pair of $(\boldsymbol{\theta}, \boldsymbol{\theta}')$ such that $p\left(\boldsymbol{X}|\boldsymbol{E}, \boldsymbol{\theta}\right) = p\left(\boldsymbol{X}|\boldsymbol{E}', \boldsymbol{\theta}'\right)$. Let $pa(j) = \{k \in V : e_{jk} = 1\}$ and $pa'(j) = \{k \in V : e'_{jk} = 1\}$ be the parent sets of node j in $\mathcal{G}$ and $\mathcal{G}'$, respectively. DAG factorization leads to the following equation for all $x_1, \ldots, x_p$,

$$\sum_{j=1}^{p} \log Pr\left(X_j = x_j \,\middle|\, \boldsymbol{X}_{pa(j)} = \boldsymbol{x}_{pa(j)}, \boldsymbol{E}, \boldsymbol{\theta}\right) = \sum_{j=1}^{p} \log Pr\left(X_j = x_j \,\middle|\, \boldsymbol{X}_{pa'(j)} = \boldsymbol{x}_{pa'(j)}, \boldsymbol{E}', \boldsymbol{\theta}'\right). \tag{1}$$

Without loss of generality, we assume that the nodes are labeled such that there is no directed edge in $\boldsymbol{E}$ from later node to earlier node. Such labeling is also known as perfect/topological ordering of DAG $\mathcal{G}$. Define a set of edges that are connected to node j and have opposite directions in $\boldsymbol{E}$ and $\boldsymbol{E}'$, $re(j) = \{k \in V : e_{jk} = e'_{kj} = 1\}$ for $j = 1, \ldots, p$. For $k \in re(j)$, $\boldsymbol{E}$ includes an edge $k \to j$, while $\boldsymbol{E}'$ has the reverse edge $j \to k$. If $re(j) = \emptyset$ for all j, there exists no pair of nodes (j, k) such that $e_{jk} = e'_{kj} = 1$. This means $\boldsymbol{E} = \boldsymbol{E}'$, because Markov equivalent DAGs have the same skeleton. We will show by mathematical induction that $re(j) = \emptyset$ for all j, which contradicts the assumption that $\boldsymbol{E} \neq \boldsymbol{E}'$.

For node p that is the last element of the perfect ordering of $\mathcal{G}$, we have $pa(p) = pa'(p) \cup re(p)$ due to the same skeleton of $\mathcal{G}$ and $\mathcal{G}'$. Taking the difference of the equality (1) at $(x_1, x_2, \ldots, x_p + 1)$ and $(x_1, x_2, \ldots, x_p)$ yields,

$$\sum_{k \in pa'(p) \cup re(p)} \beta_{pk} x_k + \gamma_p - \log(x_p + 1) = \sum_{k \in pa'(p)} \beta'_{pk} x_k + \gamma'_p - \log(x_p + 1) + \sum_{k \in re(p)} \Big[\beta'_{kp} x_k - \left\{\exp\left(\beta'_{kp}(x_p + 1) + \sum_{l \in pa'(k) \setminus \{p\}} \beta'_{kl} x_\ell + \gamma'_k\right) - \exp\left(\beta'_{kp} x_p + \sum_{l \in pa'(k) \setminus \{p\}} \beta'_{kl} x_\ell + \gamma'_k\right)\right\} + \log \frac{1 + \exp(\alpha'_{kp} x_p + \sum_{l \in pa'(k) \setminus \{p\}} \alpha'_{kl} x_\ell + \delta'_k)}{1 + \exp(\alpha'_{kp}(x_p + 1) + \sum_{l \in pa'(k) \setminus \{p\}} \alpha'_{kl} x_\ell + \delta'_k)}\Big],$$

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for $x_1, \ldots, x_p > 0$ which can be simplified as,

$$\sum_{k \in pa'(p)} \beta_{pk} x_k + \gamma_p + \sum_{k \in re(p)} \left[\beta_{pk} x_k + \exp\left(\beta'_{kp}(x_p + 1) + \sum_{l \in pa'(k) \setminus \{p\}} \beta'_{kl} x_\ell + \gamma'_k \right) \right] =$$
$$\sum_{k \in pa'(p)} \beta'_{pk} x_k + \gamma'_p + \sum_{k \in re(p)} \left[\beta'_{kp} x_k + \exp\left(\beta'_{kp} x_p + \sum_{l \in pa'(k) \setminus \{p\}} \beta'_{kl} x_\ell + \gamma'_k \right) + \right.$$
$$\left. \log \frac{1 + \exp(\alpha'_{kp} x_p + \sum_{l \in pa'(k) \setminus \{p\}} \alpha'_{kl} x_\ell + \delta'_k)}{1 + \exp(\alpha'_{kp}(x_p + 1) + \sum_{l \in pa'(k) \setminus \{p\}} \alpha'_{kl} x_\ell + \delta'_k)} \right]. \tag{2}$$

The equation (2) holds for all $x_1, \ldots, x_p > 0$ if and only if $\beta_{pk} = \alpha'_{kp} = \beta'_{kp} = 0$ for $k \in re(p)$, $\beta_{pk} = \beta'_{pk}$ for $k \in pa'(p)$, and $\gamma_p = \gamma'_p$. The fact that $\alpha'_{kp} = \beta'_{kp} = 0$ for $k \in re(p)$ implies that $re(p) = \emptyset$. We also obtain that $\delta_p = \delta'_p$, $\gamma_p = \gamma'_p$, $\alpha_{pk} = \alpha'_{pk}$, $\beta_{pk} = \beta'_{pk}$ for $k \in pa(p) = pa'(p)$.

Now assume that for any $j = m+1, \ldots, p$, it holds that $re(j) = \emptyset$, $\delta_j = \delta'_j$, $\gamma_j = \gamma'_j$, $\alpha_{jk} = \alpha'_{jk}$, and $\beta_{jk} = \beta'_{jk}$ for $k \in pa(j) = pa'(j)$. We will show that it also holds for $j = m$. Define the child sets of node m in graphs $\mathcal{G}$ and $\mathcal{G}'$ by $ch(m) = \{k \in V : e_{km} = 1\}$ and $ch'(m) = \{k \in V : e'_{km} = 1\}$, respectively. Again since $\mathcal{G}$ and $\mathcal{G}'$ share the same skeleton, $pa(m) = pa'(m) \cup re(m)$ and $ch'(m) = ch(m) \cup re(m)$. The difference of the equality (1) at $(x_1, \ldots, x_m + 1, \ldots, x_p)$ and $(x_1, \ldots, x_m, \ldots, x_p)$ is given by,

$$\sum_{k \in pa'(m) \cup re(m)} \beta_{mk} x_k + \gamma_m - \log(x_m + 1) + \sum_{k \in ch(m)} [\beta_{km} x_k -$$
$$\left\{ \exp\left(\beta_{km}(x_m + 1) + \sum_{l \in pa(k) \setminus \{m\}} \beta_{kl} x_\ell + \gamma_k \right) - \exp\left(\beta_{km} x_m + \sum_{l \in pa(k) \setminus \{m\}} \beta_{kl} x_\ell + \gamma_k \right) \right\} +$$
$$\left. \log \frac{1 + \exp(\alpha_{km} x_m + \sum_{l \in pa(k) \setminus \{m\}} \alpha_{kl} x_\ell + \kappa_k)}{1 + \exp(\alpha_{km}(x_m + 1) + \sum_{l \in pa(k) \setminus \{m\}} \alpha_{kl} x_\ell + \gamma_k)} \right] =$$
$$\sum_{k \in pa'(m)} \beta'_{mk} x_k + \gamma'_m - \log(x_m + 1) + \sum_{k \in re(m) \cup ch(m)} [\beta'_{km} x_k -$$
$$\left\{ \exp\left(\beta'_{km}(x_m + 1) + \sum_{l \in pa'(k) \setminus \{m\}} \beta'_{kl} x_\ell + \gamma'_k \right) - \exp\left(\beta'_{km} x_m + \sum_{l \in pa'(k) \setminus \{m\}} \beta'_{kl} x_\ell + \gamma'_k \right) \right\} +$$
$$\left. \log \frac{1 + \exp(\alpha'_{km} x_m + \sum_{l \in pa'(k) \setminus \{m\}} \alpha'_{kl} x_\ell + \delta'_k)}{1 + \exp(\alpha'_{km}(x_m + 1) + \sum_{l \in pa'(k) \setminus \{m\}} \alpha'_{kl} x_\ell + \delta'_k)} \right],$$

for all $x_1, \ldots, x_p > 0$. Note that $ch(m) \subset \{m+1, \ldots, p\}$. Hence, if $k \in ch(m)$, then we have $\delta_k = \delta'_k$, $\gamma_k = \gamma'_k$, $\alpha_{kl} = \alpha'_{kl}$, and $\beta_{kl} = \beta'_{kl}$ for $l \in pa(k) = pa'(k)$ due to the induction assumption. Simplifying the above equation, we obtain,

$$\sum_{k \in pa'(m)} \beta_{mk} x_k + \gamma_m + \sum_{k \in re(m)} \left[\beta_{mk} x_k + \exp\left(\beta'_{km}(x_m + 1) + \sum_{l \in pa'(k) \setminus \{m\}} \beta'_{kl} x_\ell + \gamma'_k \right) \right] =$$
$$\sum_{k \in pa'(m)} \beta'_{mk} x_k + \gamma'_m + \sum_{k \in re(m)} \left[\beta'_{km} x_k + \exp\left(\beta'_{km} x_m + \sum_{l \in pa'(k) \setminus \{m\}} \beta'_{kl} x_\ell + \gamma'_k \right) + \right.$$
$$\left. \log \frac{1 + \exp(\alpha'_{km} x_m + \sum_{l \in pa'(k) \setminus \{m\}} \alpha'_{kl} x_\ell + \delta'_k)}{1 + \exp(\alpha'_{km}(x_m + 1) + \sum_{l \in pa'(k) \setminus \{m\}} \alpha'_{kl} x_\ell + \delta'_k)} \right]. \tag{3}$$

The sufficient and necessary condition for the equation (3) to hold for all $x_1, \ldots, x_p > 0$ is that $\beta_{mk} = \alpha'_{km} = \beta'_{km} = 0$ for $k \in re(m)$, $\beta_{mk} = \beta'_{mk}$ for $k \in pa'(m)$, and $\gamma_m = \gamma'_m$. It follows that $re(m) = \emptyset$, and therefore we have $\delta_m = \delta'_m$, $\gamma_m = \gamma'_m$, $\alpha_{mk} = \alpha'_{mk}$, and $\beta_{mk} = \beta'_{mk}$ for $k \in pa(m) = pa'(m)$, which completes the proof.

B. Alternative Proof of Identifiability of Poisson Bayesian Networks

We provide an alternative proof for identifiability of Poisson BN. For Poisson BNs, the conditional distribution of each node given its parents is assumed to be Poisson distribution,

$$Pr\left(X_j = x \left| \boldsymbol{X}_{pa(j)}\right.\right) = \frac{\exp\left(-\lambda_j\right)\lambda_j^x}{x!},$$

where $\log(\lambda_j) = \sum_{k \in pa(j)} \beta_{jk} X_k + \gamma_j$. Here, Poisson BNs are parametrized by $\boldsymbol{\beta} = \{\beta_{jk}\}_{j,k}$, $\boldsymbol{\gamma} = \{\gamma_j\}_j$, and the network structure $\boldsymbol{E}$. Let $p\left(\boldsymbol{X}|\boldsymbol{E},\boldsymbol{\beta},\boldsymbol{\gamma}\right)$ and $p\left(\boldsymbol{X}|\boldsymbol{E}',\boldsymbol{\beta}',\boldsymbol{\gamma}'\right)$ denote the sampling distributions of two Markov equivalent Poisson BNs. As in the proof of Theorem 1, we will show by contradiction that

$$p\left(\boldsymbol{X}|\boldsymbol{E},\boldsymbol{\beta},\boldsymbol{\gamma}\right) \neq p\left(\boldsymbol{X}|\boldsymbol{E}',\boldsymbol{\beta}',\boldsymbol{\gamma}'\right).$$

Suppose that $p\left(\boldsymbol{X}|\boldsymbol{E},\boldsymbol{\beta},\boldsymbol{\gamma}\right) = p\left(\boldsymbol{X}|\boldsymbol{E}',\boldsymbol{\beta}',\boldsymbol{\gamma}'\right)$. Then it follows that

$$\sum_{j=1}^{p} \log Pr\left(X_j = x_j \left| \boldsymbol{X}_{pa(j)} = \boldsymbol{x}_{pa(j)}, \boldsymbol{E},\boldsymbol{\beta},\boldsymbol{\gamma}\right.\right) = \sum_{j=1}^{p} \log Pr\left(X_j = x_j \left| \boldsymbol{X}_{pa'(j)} = \boldsymbol{x}_{pa'(j)}, \boldsymbol{E}',\boldsymbol{\beta}',\boldsymbol{\gamma}'\right.\right), \tag{4}$$

for all $x_1, \ldots, x_p$. Without loss of generality, assume $(1, \ldots, p)$ to be the true causal ordering of $\boldsymbol{E}$. Recall that having $re(j) = \emptyset$ for all j implies $\boldsymbol{E} = \boldsymbol{E}'$. We will show by mathematical induction that $re(j) = \emptyset$ for all j, which contradicts the assumption $\boldsymbol{E} \neq \boldsymbol{E}'$.

For node p, we obtain $pa(p) = pa'(p) \cup re(p)$ by combining two facts that (i) there is no directed edge in $\boldsymbol{E}$ pointing away from node p and (ii) $\boldsymbol{E}$ has the same skeleton with $\boldsymbol{E}'$. The difference of the equality (4) at $(x_1, \ldots, x_p + 1)$ and $(x_1, \ldots, x_p)$ is calculated as

$$\sum_{k \in pa'(p) \cup re(p)} \beta_{pk} x_k + \gamma_p - \log(x_p + 1) = \sum_{k \in pa'(p)} \beta'_{pk} x_k + \gamma'_p - \log(x_p + 1) + \sum_{k \in re(p)} \Big[\beta'_{kp} x_k -$$
$$\left\{\exp\left(\beta'_{kp}(x_p + 1) + \sum_{l \in pa'(k)\setminus\{p\}} \beta'_{kl} x_\ell + \gamma'_k\right) - \exp\left(\beta'_{kp} x_p + \sum_{l \in pa'(k)\setminus\{p\}} \beta'_{kl} x_\ell + \gamma'_k\right)\right\}\Bigg].$$

It follows that

$$\sum_{k \in pa'(p)} \beta_{pk} x_k + \gamma_p + \sum_{k \in re(p)} \left[\beta_{pk} x_k + \exp\left(\beta'_{kp}(x_p + 1) + \sum_{l \in pa'(k)\setminus\{p\}} \beta'_{kl} x_\ell + \gamma'_k\right)\right]$$
$$= \sum_{k \in pa'(p)} \beta'_{pk} x_k + \gamma'_p + \sum_{k \in re(p)} \left[\beta'_{kp} x_k + \exp\left(\beta'_{kp} x_p + \sum_{l \in pa'(k)\setminus\{p\}} \beta'_{kl} x_\ell + \gamma'_k\right)\right] \tag{5}$$

for all $(x_1, \ldots, x_p)$. The equation (5) holds for all $(x_1, \ldots, x_p)$ if and only if $\beta_{pk} = \beta'_{kp} = 0$ for $k \in re(p)$, $\beta_{pk} = \beta'_{pk}$ for $k \in pa'(p)$, and $\gamma_p = \gamma'_p$. We can deduce $re(p) = \emptyset$ and $pa(p) = pa'(p)$ from the fact that $\beta_{pk} = \beta'_{kp} = 0$ for $k \in re(p)$.

Now assume that for any $j = m+1, \ldots, p$, it holds that $re(j) = \emptyset$, $\gamma_j = \gamma'_j$, and $\beta_{jk} = \beta'_{jk}$ for $k \in pa(j) = pa'(j)$. We will show that it also holds for $j = m$. The same skeleton of $\boldsymbol{E}$ and $\boldsymbol{E}'$ yields $pa(m) \cup ch(m) = pa'(m) \cup re(m) \cup ch(m) = pa'(m) \cup ch'(m)$. Taking the difference of the equality (4) at $(x_1, \ldots, x_m + 1, \ldots, x_p)$ and $(x_1, \ldots, x_m, \ldots, x_p)$, we get,

$$\sum_{k \in pa'(m) \cup re(m)} \beta_{mk} x_k + \gamma_m - \log(x_m + 1) + \sum_{k \in ch(m)} \Big[\beta_{km} x_k -$$
$$\left\{\exp\left(\beta_{km}(x_m + 1) + \sum_{l \in pa(k)\setminus\{m\}} \beta_{kl} x_\ell + \gamma_k\right) - \exp\left(\beta_{km} x_m + \sum_{l \in pa(k)\setminus\{m\}} \beta_{kl} x_\ell + \gamma_k\right)\right\}\Bigg]$$
$$= \sum_{k \in pa'(m)} \beta'_{mk} x_k + \gamma'_m - \log(x_m + 1) + \sum_{k \in re(m) \cup ch(m)} \Big[\beta'_{km} x_k -$$
$$\left\{\exp\left(\beta'_{km}(x_m + 1) + \sum_{l \in pa'(k)\setminus\{m\}} \beta'_{kl} x_\ell + \gamma'_k\right) - \exp\left(\beta'_{km} x_m + \sum_{l \in pa'(k)\setminus\{m\}} \beta'_{kl} x_\ell + \gamma'_k\right)\right\}\Bigg],$$

for all $(x_1, \dots, x_p)$. Since $ch(m) \subset \{m+1, \dots, p\}$, we conclude that for all $k \in ch(m)$, $\gamma_k = \gamma'_k$ and $\beta_{kl} = \beta'_{kl}$ for $l \in pa(k) = pa'(k)$. Hence we can simplify the above equation as,

$$\begin{aligned} &\sum_{k \in pa'(m)} \beta_{mk} x_k + \gamma_m + \sum_{k \in re(m)} \left[\beta_{mk} x_k + \exp\left(\beta'_{km}(x_m + 1) + \sum_{l \in pa'(k) \setminus \{m\}} \beta'_{kl} x_\ell + \gamma'_k \right) \right] \\ &= \sum_{k \in pa'(m)} \beta'_{mk} x_k + \gamma'_m + \sum_{k \in re(m)} \left[\beta'_{km} x_k + \exp\left(\beta'_{km} x_m + \sum_{l \in pa'(k) \setminus \{m\}} \beta'_{kl} x_\ell + \gamma'_k \right) \right]. \end{aligned} \tag{6}$$

The equation (6) holds for all $(x_1, \dots, x_p)$ if and only if $\beta_{mk} = \beta'_{km} = 0$ for $k \in re(m)$, $\beta_{mk} = \beta'_{mk}$ for $k \in pa'(m)$, and $\gamma_m = \gamma'_m$. Therefore, we obtain $re(m) = \emptyset$ and $pa(m) = pa'(m)$, which completes the proof. Finally, we remark that in Park & Raskutti (2015), their proof exploited the overdispersion properties of Poisson BNs whereas our proof does not rely on that specific property and is therefore generalizable to other distribution families such as negative binomial and continuous distributions (results not shown).

C. Analytical Integration of Equations (3)-(5)

We integrate out ρ from $p(\boldsymbol{E}, \rho)$,

$$\begin{aligned} p(\boldsymbol{E}) &= \int p(\boldsymbol{E}|\rho) p(\rho) d\rho \\ &\propto \int z(\rho)^{-1} \prod_{j \neq k} \rho^{e_{jk}} (1-\rho)^{1-e_{jk}} I(\mathcal{G} \in \mathcal{D}) z(\rho) \rho^{a_\rho - 1} (1-\rho)^{b_\rho - 1} d\rho \\ &= I(\mathcal{G} \in \mathcal{D}) \int \rho^{\sum_{j \neq k} e_{jk} + a_\rho - 1} (1-\rho)^{\sum_{j \neq k} (1 - e_{jk}) + b_\rho - 1} d\rho \\ &= B\left(\sum_{j \neq k} e_{jk} + a_\rho, \sum_{j \neq k} (1 - e_{jk}) + b_\rho \right) I(\mathcal{G} \in \mathcal{D}), \end{aligned}$$

where the last equality holds because the integrand is the kernel of a beta distribution.

D. More Details on the Real Data Analyses

Collection Process of AhR-Knockout scRNA-seq Data. The scRNA-seq experiments were performed on five mice with AhR knockout targeted to intestinal stem cells. AhR is a ligand-activated transcription factor that is capable of integrating external environmental stimuli and host responses to modulate intestinal stem cell development, tissue regeneration, and colon cancer risk (Kim *et al.*, 2016; Safe *et al.*, 2018). We use this dataset to construct gene regulatory networks for genes that belong to Wnt signaling pathway. On average each mouse contributed ∼6,000 cells.

List of Pairs of Transcription Factors and Targets. We provide the complete list of pairs of transcription factors and targets used in Section 3.2.1 in Tables T1 and T2.

List of Genes in Wnt Pathway. We have used following genes in Section 3.2.2: Axin1, Cby1, Ccnd1, Chd8, Cacybp, Crebbp, Csnk1a1, Csnk1e, Csnk2a1, Ctbp1, Ctnnbip1, Ctnnb1, Cul1, Daam1, Dvl1, Frat1, Fbxw11, Gsk3b, Jun, Mapk8, Map3k7, Myc, Nlk, Senp2, Ppard, Ppp3ca, Prkaca, Prkca, Psen1, Rac1, Rbx1, Rhoa, Rock2, Ruvbl1, Siah1a, Skp1a, Smad3, Smad4, Tbl1x, and Trp53.

Data Preprocessing using Seurat Package. Cells of the same type were used to recover the gene regulatory network in Section 3.2.1. To identify the cell type, we clustered data from one mouse by using the scRNA-seq clustering algorithm in `R` package `Seurat` (Butler *et al.*, 2018). Before applying the algorithm, we normalized the data with the LogNormalize method and selected $2,000$ genes that showed high cell-to-cell variation. The principal component analysis was done on the scaled data to calculate the distance between cells for the clustering analysis. We considered the first 19 principal components to build the cellular distance matrix. The clustering algorithm in `Seurat` was applied to construct a K-nearest neighborhood graph based on the distance matrix and partition it into several clusters. The resolution parameter of 0.1 was specified, resulting in 9 cell types in total. We chose the cell type that includes $n = 1,025$ cells with 60% zeros, which is comparable to our simulation study in Section 3.1.

References

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T1. Pairs of Transcription Factors (TF) and Targets (TG).

TF	TG	TF	TG	TF	TG	TF	TG	TF	TG
Aes	Ihh	Egr1	Ccnd2	Fosb	Cdh1	Jun	Odc1	Nfkb1	Akt1
Apc	Ctnnb1	Egr1	Crebbp	Fosb	Fos	Jun	Plin2	Nfkb1	Xiap
Apc	Tcf4	Egr1	Eno1	Foxa1	Cbx5	Jun	Nectin2	Nfkb1	Xiap
Apc	Trp53	Egr1	Hsd11b2	Foxa1	Muc2	Jun	Rab18	Nfkb1	Ccnd1
Apex1	Nfkb1	Egr1	Id3	Foxa3	Cdx2	Jun	Serpinb5	Nfkb1	Ccnd1
Apex1	Nfkb1	Egr1	Ifngr1	Fus	Sod2	Jun	Tpr	Nfkb1	Cd38
Arid5b	Sox9	Egr1	Jun	Gata6	Cdx2	Jun	Trp53	Nfkb1	Cdk4
Atf4	Hspa5	Egr1	Jund	Gtf2i	Ccnd1	Jun	Trp53	Nfkb1	Ctnnb1
Atf4	Ihh	Egr1	Map1lc3b	Gtf2i	Cfdp1	Junb	Ccnd1	Nfkb1	Cxadr
Cdx1	Cdx2	Egr1	Ppp1r1b	Gtf2i	Csrp2	Jund	Jun	Nfkb1	Ebna1bp2
Cdx2	Cdh17	Egr1	Pten	Gtf2i	Fos	Keap1	Nfe2l2	Nfkb1	Elf3
Cdx2	Cdx1	Egr1	Pten	Gtf2i	Fos	Khdrbs1	Ece1	Nfkb1	Kdm2a
Cdx2	Glb1	Egr1	Ptges2	Gtf2i	Hspa5	Klf4	Ccnd2	Nfkb1	Fos
Cdx2	Kitl	Egr1	Smad7	Gtf2i	Id1	Klf4	Cdh1	Nfkb1	Fth1
Cdx2	Mgam	Egr1	Stim1	Gtf2i	Nsd1	Klf5	Egr1	Nfkb1	Gclm
Cebpz	Hspa1b	Egr1	Trp53	Hdac1	Jun	Klf6	Asah1	Nfkb1	Glrx
Chd4	Jun	Ehmt2	Ccnd1	Hdac1	Nfkbia	Klf6	Pttg1	Nfkb1	Hes1
Chd7	Sox4	Ehmt2	Pparg	Hdac1	Pparg	Mapk1	Egr1	Nfkb1	Hmgcr
Chd8	Stat3	Ep300	Ccnd1	Hdac1	Ppp2ca	Max	Bax	Nfkb1	Hsd11b2
Chd8	Tcf4	Ep300	Ccnd1	Hdac1	Sp1	Max	Odc1	Nfkb1	Hsd11b2
Cited2	Pparg	Ep300	Clock	Hdac1	Epcam	Max	Trp53	Nfkb1	Junb
Clock	Bhlhe40	Ep300	Fasn	Hdac2	Nfkbia	Mdm4	Trp53	Nfkb1	Klf5
Clock	Bhlhe40	Ep300	Fos	Hdac2	Smarca4	Mdm4	Trp53	Nfkb1	Mbp
Creb3	Herpud1	Ep300	Irf2	Hdac3	Fos	Med1	Pparg	Nfkb1	Myb
Crebbp	Clock	Ep300	Klf5	Hdac3	Jun	Med1	Pparg	Nfkb1	Nfkbia
Crebbp	Ep300	Ep300	Ldha	Hdac3	Nfkbia	Myb	Gstm2	Nfkb1	Pla2g4a
Crebbp	Fos	Ep300	Mt1	Hes6	Hes1	Myb	H2afz	Nfkb1	Plin2
Crebbp	Fosb	Ep300	Myb	Hnf4a	Cdx2	Myb	Ppm1a	Nfkb1	Psme2
Crebbp	Fth1	Ep300	Pparg	Hnf4a	Foxa1	Nab1	Egr1	Nfkb1	Pten
Crebbp	Ldha	Ep300	Tapbp	Hnf4a	Gstp1	Nab1	Ifngr1	Nfkb1	Sod2
Crebbp	Med1	Ep300	Txnip	Hnf4a	Perp	Ncor1	Hdac3	Nfkb1	Sox9
Crebbp	Mt1	Esrra	Acadm	Id3	Smarcc1	Ncor1	Hes1	Nfkb1	Tapbp
Crebbp	Nfkb1	Esrra	Nrip1	Irf3	Pura	Ncor1	Nfkbia	Nfkb1	Trp53
Crebbp	Trp53	Esrra	Rb1cc1	Irf3	Socs2	Nfat5	Akr1b3	Nfkb1	Ube2h
Crebbp	Trp53	Esrra	Trp53	Irf8	Jak1	Nfat5	Bax	Nfkb1	Yy1
Ctnnb1	Casp3	Ets2	Egr1	Jun	Asah1	Nfe2l2	Akr1b3	Nfkb1	Ccnd1
Ctnnb1	Ccnd1	Ets2	Hmox2	Jun	B4galnt1	Nfe2l2	Cox17	Nfkb1	Trp53
Ctnnb1	Ccnd1	Ets2	Jun	Jun	Cat	Nfe2l2	Fos	Nfkb1	Cdh1
Ctnnb1	Cdh1	Ets2	Krt8	Jun	Ccnd1	Nfe2l2	Gclc	Nfkbia	Nfkb1
Ctnnb1	Cdx1	Ezh2	Bad	Jun	Ccnd1	Nfe2l2	Keap1	Nr1h2	Fasn
Ctnnb1	Cdx2	Ezh2	Creb3l1	Jun	Ccnd1	Nfe2l2	Keap1	Nr3c1	Cdk4
Ctnnb1	Fgfbp1	Ezh2	Trp53	Jun	Ccnd2	Nfe2l2	Mgst1	Nr3c1	Cdk6
Ctnnb1	Glul	Fos	Acp5	Jun	Fos	Nfe2l2	Mt1	Nr3c1	Hes1
Ctnnb1	Ihh	Fos	Egr1	Jun	Fos	Nfe2l2	Nfkb1	Nr3c1	Nfkbia
Ctnnb1	Ppard	Fos	H2-K1	Jun	Gclc	Nfe2l2	Pparg	Nr3c1	Vdr
Ctnnb1	Smc3	Fos	Hmgcr	Jun	Hmgcr	Nfe2l2	Pparg	Nr5a2	Ccnd1
Ctnnb1	Tcf4	Fos	Jun	Jun	Krt18	Nfe2l2	Prdx1	Nr5a2	Kras
Cux1	Dctn6	Fos	Krt18	Jun	Nfe2l2	Nfe2l2	Prdx1	Nupr1	Ep300
Ddx54	Mbp	Fos	Mt1	Jun	Nfe2l2	Nfe2l2	Prdx2	Pbx1	Cdh1
Dnajc2	Id1	Fos	Nfe2l2	Jun	Nfkb1	Nfe2l2	Sdha	Pcbp2	Trp53
Egr1	Bax	Fos	Nr3c1	Jun	Nfkb1	Nfe2l2	Sod2	Pnn	Cdx2
Egr1	Cat	Fos	Stim1	Jun	Nr3c1	Nfe2l2	Txnip	Ppard	Insig1
Egr1	Ccnd1	Fos	Trp53	Jun	Nr3c1	Nfe2l2	Txnrd1	Ppard	Pparg

T2. Pairs of Transcription Factors (TF) and Targets (TG).

TF	TG	TF	TG	TF	TG	TF	TG	TF	TG
Ppard	Sp1	Sp1	Cops5	Sp3	Rest	Trp53	Epcam	Zbtb7a	Creb3
Pparg	Atp2a2	Sp1	Cox17	Sp3	Sdc1	Trp53	Fos	Zfp36	Nfkb1
Pparg	Cat	Sp1	Cox4i1	Sp3	Sp1	Trp53	Fos		
Pparg	Ccnd1	Sp1	Csrp2	Sp3	Sp1	Trp53	Gsk3b		
Pparg	Cops5	Sp1	Cxadr	Sp3	Tspo	Trp53	Hdgf		
Pparg	Dbi	Sp1	Egr1	Sp3	Ucp2	Trp53	Hras		
Pparg	Egr1	Sp1	Esrra	Srebf1	Acss2	Trp53	Hras		
Pparg	Insr	Sp1	Fos	Srebf1	Fasn	Trp53	Hspa1b		
Pparg	Klf5	Sp1	Gstk1	Srebf1	Fasn	Trp53	Insr		
Pparg	Plin2	Sp1	Hmgcl	Srebf1	Hmgcr	Trp53	Krt19		
Pparg	Sod2	Sp1	Hmgcr	Srebf2	Acss2	Trp53	Mapk1		
Pparg	Trp53	Sp1	Hspa1b	Srebf2	Hmgcr	Trp53	Mcts1		
Pparg	Ucp2	Sp1	Hspa1b	Srebf2	Hmgcs1	Trp53	Mt1		
Ppargc1b	Esrra	Sp1	Insr	Srebf2	Hnf4a	Trp53	Nfkb1		
Pttg1	Pttg1ip	Sp1	Itga6	Stat3	Akt1	Trp53	Nr3c1		
Pura	Mbp	Sp1	Jun	Stat3	Ccnd1	Trp53	Nupr1		
Raf1	Mapk1	Sp1	Jund	Stat3	Ccnd2	Trp53	Pcbp2		
Rala	Fos	Sp1	Kdm5a	Stat3	Egr1	Trp53	Perp		
Rest	Kcnn4	Sp1	Lmna	Stat3	Fos	Trp53	Pla2g16		
Satb2	Upf3b	Sp1	Mapk1	Stat3	Fos	Trp53	Prnp		
Sf1	Sox9	Sp1	Mbp	Stat3	Klf4	Trp53	Psen1		
Smad1	Id3	Sp1	Mgst1	Stat3	Mt1	Trp53	Psme3		
Smad4	Cdh1	Sp1	Mif	Stat3	Phb	Trp53	Pten		
Smad4	Cldn3	Sp1	Myb	Stat3	Pik3r1	Trp53	Rchy1		
Smad4	Cxadr	Sp1	Myb	Stat3	Pnp	Trp53	Reep5		
Smad4	Id3	Sp1	Nr1h2	Stat3	Pten	Trp53	Rps6ka1		
Smad4	Jun	Sp1	Pla2g16	Stat3	Srebf1	Trp53	Serpinb5		
Smad4	Ocln	Sp1	Ppm1a	Stat3	Tcf4	Trp53	Siva1		
Smad4	Pparg	Sp1	Prdx6	Stat3	Trp53	Trp53	Slc6a6		
Smad4	Smad7	Sp1	Prkra	Suz12	Cdh1	Trp53	Sod2		
Smad7	Glb1	Sp1	Ptges2	Tardbp	Ran	Trp53	Srebf1		
Smad7	Nfkbia	Sp1	Nectin2	Tcf12	Smarcc1	Trp53	Tcf4		
Smarca4	Calm1	Sp1	Rest	Tcf4	Cdh1	Trp53	Tcf7l2		
Smarca4	Cdh1	Sp1	Slc3a2	Tcf4	Cdx2	Trp53	Tpt1		
Smarca4	Nfkbia	Sp1	Slc5a1	Tcf4	Cops5	Trp53	Tpt1		
Smarca4	Rest	Sp1	Slc5a1	Tcf4	Id3	Trp53	Tyms		
Sox4	Ctnnb1	Sp1	Smad7	Tcf4	Smc3	Trp53	Klf4		
Sox4	Ezh2	Sp1	Smarcc1	Tcf7l2	Cdx1	Ube2k	Nfe2l2		
Sox4	Mecom	Sp1	Sod2	Thrap3	Pparg	Vdr	Ctnnb1		
Sox4	Tcf4	Sp1	Srebf1	Tob1	Ccnd1	Vezf1	Morf4l1		
Sox9	Cdh1	Sp1	Srebf1	Trim28	Sox9	Vezf1	Stmn1		
Sox9	Ctnnb1	Sp1	Sult1a1	Trp53	Bak1	Xbp1	Fasn		
Sp1	Abcb1a	Sp1	Tpt1	Trp53	Bax	Xbp1	Herpud1		
Sp1	App	Sp1	Tpt1	Trp53	Bax	Xbp1	Tmbim6		
Sp1	Asah1	Sp1	Tspo	Trp53	Btg2	Ybx1	Abcb1a		
Sp1	Bhlhe40	Sp1	Tyms	Trp53	Casp3	Ybx1	Nono		
Sp1	Bsg	Sp1	Ucp2	Trp53	Casp6	Yy1	Cdh1		
Sp1	Ccnd1	Sp3	Bsg	Trp53	Casp7	Yy1	Fos		
Sp1	Ccnd1	Sp3	Itga6	Trp53	Casp8	Yy1	Hspa5		
Sp1	Ccnd2	Sp3	Lmna	Trp53	Ddit4	Yy1	Nr3c1		
Sp1	Cdk6	Sp3	Mapk1	Trp53	Ddit4	Yy1	Tcf4		
Sp1	Ces1d	Sp3	Pla2g16	Trp53	Egr1	Yy1	Trp53		
Sp1	Cirbp	Sp3	Prkra	Trp53	Ei24	Zbtb20	Sox9		