Implicit Distributional Reinforcement Learning: Appendix

A Proof of Lemma 1

Denote

$$\mathcal{H} = \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s})} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s}),$$

and

$$\mathcal{H}_L = \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(0)})} \log \frac{1}{L+1}\sum_{\ell=0}^{L} \pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(\ell)}),$$

and

$$\pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)}) = \frac{1}{L+1}\sum_{\ell=0}^{L} \pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(\ell)}).$$

Notice that $\boldsymbol{\xi}$s are from the same distribution, so we have

$$\begin{aligned}\mathcal{H}_L &= \frac{1}{L+1}\sum_{i=0}^{L} \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(i)})} \log \frac{1}{L+1}\sum_{\ell=0}^{L} \pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(\ell)}) \\ &= \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)}).\end{aligned}$$

Use the identity that $\mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s})} = \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})}$, we can rewrite $\mathcal{H}$ as

$$\mathcal{H} = \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s}).$$

Therefore, we have

$$\begin{aligned}\mathcal{H}_L - \mathcal{H} &= \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})} \log \frac{\pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})}{\pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s})} \\ &= \mathrm{KL}(\pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})||\pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s})) \geq 0.\end{aligned}$$

To compare between $\mathcal{H}_L$ and $\mathcal{H}_{L+1}$, rewrite $\mathcal{H}_L$ as

$$\mathcal{H}_L = \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L),(L+1)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})$$

and $\mathcal{H}_{L+1}$ as

$$\begin{aligned}\mathcal{H}_{L+1} &= \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L),(L+1)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(0)})} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L+1)}) \\ &= \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L),(L+1)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L+1)})\end{aligned}$$

and the difference will be

$$\begin{aligned}\mathcal{H}_L - \mathcal{H}_{L+1} &= \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L),(L+1)}\sim p(\boldsymbol{\xi})} \mathbb{E}_{\boldsymbol{a}\sim\pi_{\boldsymbol{\theta}}(\boldsymbol{a}\,|\,\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})} \big[\log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)}) - \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L+1)})\big] \\ &= \mathbb{E}_{\boldsymbol{\xi}^{(0),..,(L),(L+1)}\sim p(\boldsymbol{\xi})} \mathrm{KL}(\pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L)})||\pi_{\boldsymbol{\theta}}(\boldsymbol{a}|\boldsymbol{s},\boldsymbol{\xi}^{(0):(L+1)})) \geq 0.\end{aligned}$$

Finally, we arrive at the conclusion that for any ℓ, we have

$$\mathcal{H} \leq \mathcal{H}_{\ell+1} \leq \mathcal{H}_{\ell}.$$

B Detailed pseudo code

Algorithm 2 Implicit Distributional Actor-Critic (IDAC)

Require: Learning rate λ, batch size M, quantile number K, action number J and noise number L, target entropy $\mathcal{H}_t$.
Initial policy network parameter $\boldsymbol{\theta}$, action-value function network parameter $\boldsymbol{\omega}_1$,$\boldsymbol{\omega}_2$, entropy parameter η.
Initial target network parameter $\tilde{\boldsymbol{\omega}}_1 = \boldsymbol{\omega}_1$,$\tilde{\boldsymbol{\omega}}_2 = \boldsymbol{\omega}_2$.
for the number of environment steps **do**
Sample M number of transitions $\{\boldsymbol{s}_t^i, \boldsymbol{a}_t^i, r_t^i, \boldsymbol{s}_{t+1}^i\}_{i=1}^M$ from the replay buffer
Sample $\boldsymbol{\epsilon}_t^{i,(k)}, \boldsymbol{\epsilon}_{t+1}^{i,(k)}, \boldsymbol{\xi}_{t+1}^{i,(\ell)}$ from $\mathcal{N}(\mathbf{0}, \mathbf{I})$ for $i = 1 \cdots M$ and $k = 1 \cdots K$ and $\ell = 0 \cdots L$.
Sample $\boldsymbol{a}_{t+1}^i \sim \pi_{\boldsymbol{\theta}}(\cdot \,|\, \boldsymbol{s}_{t+1}^i, \boldsymbol{\xi}_{t+1}^{i,(0)}) = \mathcal{N}(\mathcal{T}_{\boldsymbol{\theta}}^1(\boldsymbol{s}_{t+1}^i, \boldsymbol{\xi}_{t+1}^{i,(0)}), \mathcal{T}_{\boldsymbol{\theta}}^2(\boldsymbol{s}_{t+1}^i, \boldsymbol{\xi}_{t+1}^{i,(0)}))$ for $i = 1 \cdots M$.

Apply Bellman update to create samples (of return distribution)

$$y_{1,i,k} = r_t^i + \gamma G_{\tilde{\boldsymbol{\omega}}_1}(\boldsymbol{s}_{t+1}^i, \boldsymbol{a}_{t+1}^i, \boldsymbol{\epsilon}_{t+1}^{i,(k)}) \quad \text{\# Calculate target values}$$
$$y_{2,i,k} = r_t^i + \gamma G_{\tilde{\boldsymbol{\omega}}_2}(\boldsymbol{s}_{t+1}^i, \boldsymbol{a}_{t+1}^i, \boldsymbol{\epsilon}_{t+1}^{i,(k)}) \quad \text{\# Calculate target values}$$

and let

$$(\overrightarrow{y}_{1,i,1}, \dots, \overrightarrow{y}_{1,i,K}) = \text{StopGradient}(\text{sort}(y_{1,i,1}, \dots, y_{1,i,K})) \quad \text{\# Obtain target quantile estimation}$$
$$(\overrightarrow{y}_{2,i,1}, \dots, \overrightarrow{y}_{2,i,K}) = \text{StopGradient}(\text{sort}(y_{2,i,1}, \dots, y_{2,i,K})) \quad \text{\# Obtain target quantile estimation}$$
$$\overrightarrow{y}_{i,k} = \min(\overrightarrow{y}_{1,i,k}, \overrightarrow{y}_{2,i,k}), \text{ for } i = 1 \cdots M; k = 1 \cdots K$$

Generate samples $x_{1,i,k} = G_{\boldsymbol{\omega}_1}(\boldsymbol{s}_t^i, \boldsymbol{a}_t^i, \boldsymbol{\epsilon}_t^{i,(k)})$ and $x_{2,i,k} = G_{\boldsymbol{\omega}_2}(\boldsymbol{s}_t^i, \boldsymbol{a}_t^i, \boldsymbol{\epsilon}_t^{i,(k)})$, and let

$$(\overrightarrow{x}_{1,i,1}, \dots, \overrightarrow{x}_{1,i,K}) = \text{sort}(x_{1,i,1}, \dots, x_{1,i,K})$$
$$(\overrightarrow{x}_{2,i,1}, \dots, \overrightarrow{x}_{2,i,K}) = \text{sort}(x_{2,i,1}, \dots, x_{2,i,K})$$

Update action-value function parameter $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ by minimizing the quantile loss

$$J(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = \frac{1}{M}\sum_{i=1}^M \frac{1}{K^2}\sum_{k=1}^K\sum_{k'=1}^K \rho_{\tau_k}^{\kappa}(\overrightarrow{y}_{i,k} - \overrightarrow{x}_{1,i,k'}) + \frac{1}{M}\sum_{i=1}^M \frac{1}{K^2}\sum_{k=1}^K\sum_{k'=1}^K \rho_{\tau_k}^{\kappa}(\overrightarrow{y}_{i,k} - \overrightarrow{x}_{2,i,k'}).$$

Sample $\Xi_t^{i,h}$, $\boldsymbol{\epsilon}_t^{i,(j)}$ from $\mathcal{N}(\mathbf{0}, \mathbf{I})$, for $i = 1 \cdots M, j = 1 \cdots J$ and $h = 0 \cdots L + J$, and form $\boldsymbol{\xi}_t^{i,(j,\ell)}$ from $\Xi_t^{i,h}$ by concatenating L of them to the rest of Js. Sample $\boldsymbol{a}_t^{i,(j)} \sim \pi_{\boldsymbol{\theta}}(\cdot \,|\, \boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(j,0)}) = \mathcal{N}(\mathcal{T}_{\boldsymbol{\theta}}^1(\boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(j,0)}), \mathcal{T}_{\boldsymbol{\theta}}^2(\boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(j,0)}))$ using

$$\boldsymbol{a}_t^{i,(j)} = \mathcal{T}_{\boldsymbol{\theta}}(\boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(j,0)}, \boldsymbol{e}_t^i) = \mathcal{T}_{\boldsymbol{\theta}}^1(\boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(j,0)}) + \boldsymbol{e}_t^i \odot \mathcal{T}_{\boldsymbol{\theta}}^2(\boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(j,0)}),\ \boldsymbol{e}_t^i \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

for $i = 1, \cdots, M$.
Update the policy function parameter $\boldsymbol{\theta}$ by minimizing

$$J(\boldsymbol{\theta}) = -\frac{1}{M}\sum_{i=1}^M \left\{ \frac{1}{2J}\sum_{z=1}^2\sum_{j=1}^J G_{\boldsymbol{\omega}_z}(\boldsymbol{s}_t^i, \boldsymbol{a}_t^{i,(j)}, \boldsymbol{\epsilon}_t^{i,(j)}) - \exp(\eta)\sum_{j=1}^J \frac{1}{J}\Big[\log \frac{\sum_{\ell=0}^L \pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t^{i,(j)} | \boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(j,\ell)})}{L+1}\Big] \right\}.$$

We also use stop gradient on $(\mathcal{T}_{\boldsymbol{\theta}}^1(\boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(j)}), \mathcal{T}_{\boldsymbol{\theta}}^2(\boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(j)}))$ to reduce variance on gradient as mentioned in Eq (16).
Update the log entropy parameter η by minimizing

$$J(\eta) = \frac{1}{M}\sum_{i=1}^M [\text{StopGradient}(-\log \frac{\sum_{\ell=0}^L \pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t^{i,(0)} | \boldsymbol{s}_t^i, \boldsymbol{\xi}_t^{i,(\ell)})}{L+1} - \mathcal{H}_t)\eta]$$

end for

C Hyperparameters of IDAC

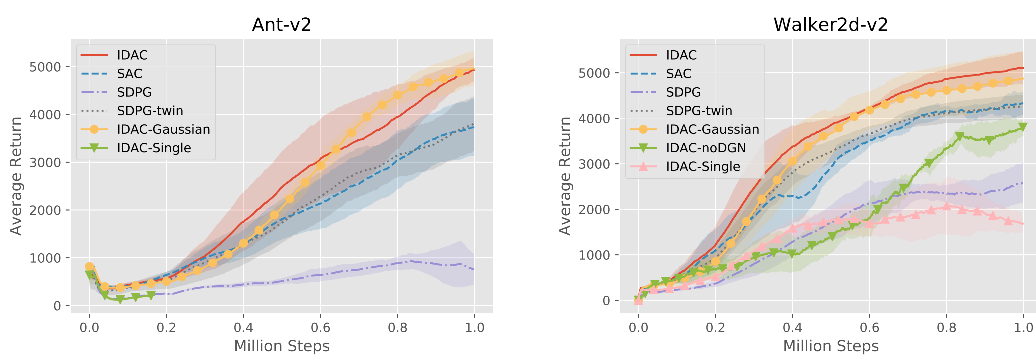
Table 3: IDAC hyperparameters

Parameter	Value
Optimizer	Adam
learning rate	3e-4
discount	0.99
replay buffer size	10^6
number of hidden layers (all networks)	2
number of hidden units per layer	256
number of samples per minibatch	256
entropy target	$-\dim(\mathcal{A})$ (*e.g.*, -6 for HalfCheetah-v2)
nonlinearity	ReLU
target smoothing coefficient	0.005
target update interval	1
gradient steps	1
distribution of $\boldsymbol{\xi}$	$\mathcal{N}(\mathbf{0}, \mathbf{I}_5)$
distribution of $\boldsymbol{\epsilon}$	$\mathcal{N}(\mathbf{0}, \mathbf{I}_5)$
J	51
K	51
L	21

D Additional ablation study

Additional ablation studies on Ant is shown in Fig. 4a for a thorough comparison. In Ant, the performance of IDAC is on par with that of IDAC-Gaussian, which outperforms the other variants.

Furthermore, we would like to learn the interaction between DGN and SIA by running ablation studies by holding each of them as a control factor; we conduct the corresponding experiments on Walker2d. From Fig. 4b, we can observe that by removing either SIA (resulting in IDAC-Gaussian) or DGN (resulting in IDAC-noDGN) from IDAC in general negatively impacts its performance, which echoes our motivation that we integrate DGN and SIA to allow them to help strengthen each other: *(i)* Modeling G exploits distributional information to help better estimate its mean Q (note C51, which outperforms DQN by exploiting distributional information, also conducts its argmax operation on Q); *(ii)* A more flexible policy may become more necessary given a better estimated Q.



(a) Additional ablation study on Ant

(b) Additional ablation study on Walker2d

Figure 4: Additional plots for ablation study

E Additional comparison with SDPG

In Fig. 5, we include a thorough comparison with SDPG (implemented based on the *stable baselines* codebase).

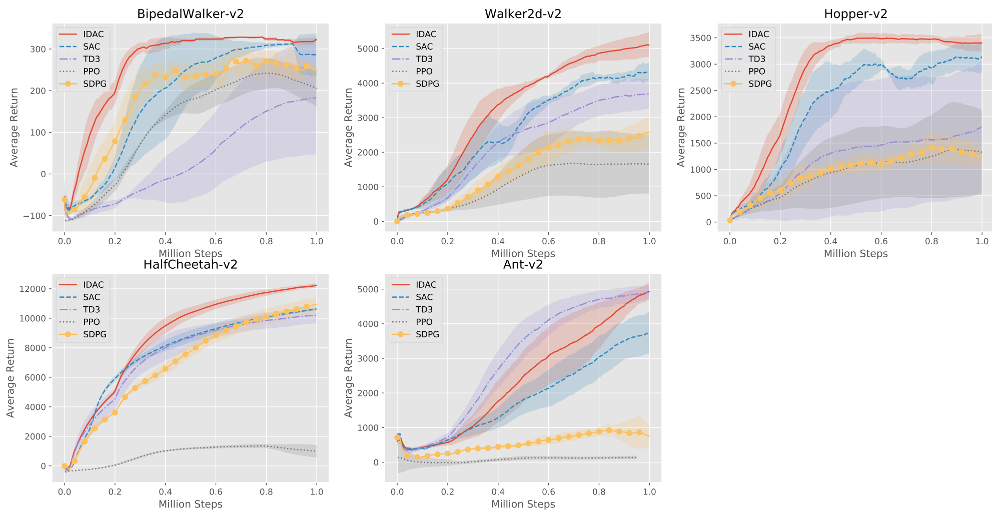


Figure 5: Training curves on continuous control benchmarks. The solid line is the average performance over 4 seeds with $\pm$ 1 std shaded, and with a smoothing window of length 100.

F Late stage policy visualization

We show in Fig. 6 the visualization of the late stage policy of one seed from Walker2d-v2 environment. We can see that SIA does provide a more flexible policy even in the late stage, where the correlations between action dimensions is clear on plot, and the marginal distributions are more flexible than a Gaussian distribution. Moreover, we randomly choose 1000 states and conduct normality tests as well as correlation tests on them. As a result, **all** of the tests indicate that the SIA policy captures the non-zero correlations between the selected dimensions, and the marginal distributions for each dimension across different states are significantly non-normal.

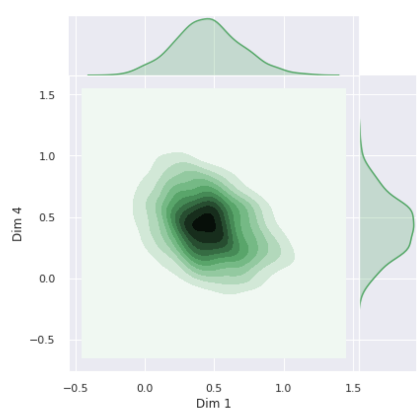


Figure 6: Visualization of the SIA on Walker2d-v2. The density contour of 1000 randomly sampled actions at a late training stage, where the x- and y-axis correspond to dimensions 1 and 4, respectively.