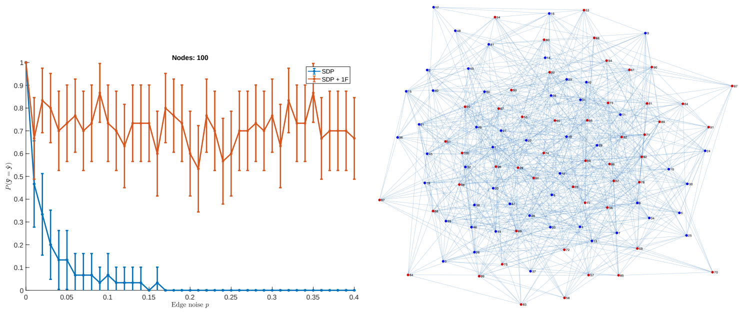
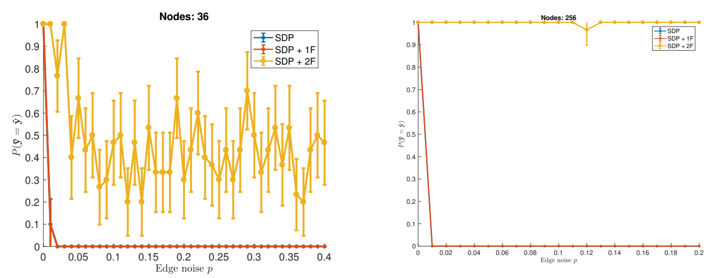
We thank all reviewers for their comments and feedback. In what follows we try to address the main concerns, we encourage reviewers to read all our responses as some are related.

R1: We thank R1 for appreciating our results and efforts put in our work. • Our theory does not make any assumption on the underlying graph except that it is connected. Thus, Theorem 1 holds for grids of different sizes or graphs other than grids. In the next figure, we sampled an Erdős-Rényi graph (used in analysis of social networks, e.g., "Random graph models of social networks" by Newman et al. (2002)) of 100 nodes with edge probability of 0.1382 ($^{3\log n}/_{n}$). The rest of the setting follows that of Section 4.2. As discussed in Section 4.1, we have that $\Delta > 0$ with high probability, and we can observe in the left plot how the addition of a single constraint boosts the probability of exact recovery. The right plot shows the sampled graph with red and blue nodes indicating the true -1 and $+1$ labels respectively. • One intuitive way to understand the improvement is by looking at eq.(7) in page 5. Without the constraint, the problem is equivalent to that of [8]. However, when the linear constraint is added, in the dual problem of the SDP relaxation, it appears a new term $\boldsymbol{N}$ that is a PSD matrix, hence, it cannot lower the value of $\lambda_2(\mathbb{E}[\boldsymbol{M}])$. We will add a line in the manuscript explaining this.



R2: • The only part we borrowed from [8] is the lower bound on $\lambda_2(\mathbb{E}[\boldsymbol{M}])$ and the setting of the dual variable $\boldsymbol{V}$ from [1] and [8]. The remaining part of our work (Lemma 1, adaption of Lemma 1 into Theorem 1, Corollary 1, Discussion on connections to eigenvalue gap in the Laplacian matrix and Fiedler vector, and Empirical evidence) is novel as noted by R1, R3. • We disagree that the overall content is difficult to follow. R1, R3 and R4 stated that the presentation was mostly clear. • Regarding the meaning of the constraint, let us provide an example how it can be interpreted as imposing demographic parity at *inference* time. Let the nodes in the graph represent individuals, where the label indicates the community a person belongs to. Then, let $\boldsymbol{a}$ be a vector of some resources that ideally should be split equally to both communities. The constraint can be interpreted as forcing a labeling to create two communities where the sum of resources is equal for each community. Finally, we also argue that even if the constraint is seen as side information, it does **not** imply that the combinatorial problem is easier, in fact, as discussed in Remark 1, it is still NP-hard in general. We can empirically corroborate that in some scenarios the addition of a **single** constraint does not improve the probability of exact recovery. In the next figure, we follow a same setting to that of Section 4.2, where now the graphs are grids of 6x6 (left) and 16x16 (right). We observe how the addition of a single constraint does not help exact recovery as suggested by our discussion on $\Delta = 0$ for square grids, however, the addition of two constraints can help square grids to be recovered exactly.



R3: • Regarding the major concern, we confirm that ρ was set to be an arbitrary finite number, in this case $-n$. Consider for a moment that we leave ρ unset, then in eq.(7) the goal would be to find a lower bound to $\lambda_2(\mathbb{E}[\boldsymbol{M}] - \rho\boldsymbol{N})$. Given that $\mathbb{E}[\boldsymbol{M}]$ and $\boldsymbol{N}$ are PSD matrices, then intuitively the optimal setting for ρ would be $-\infty$ as it would maximize the increase in $\lambda_2(\mathbb{E}[\boldsymbol{M}] - \rho\boldsymbol{N})$. However, computationally speaking, one can note that such assignment will never happen. Instead, the SDP solver will try to set ρ a **finite** value as low as possible as to observe $\lambda_2(\boldsymbol{\Lambda}) > 0$. This would be equivalent to fix ρ and let the Fiedler vector $\boldsymbol{\pi}_2$ scale as to maximize ϵ_1. For example, let the Fiedler vector have a norm of $\sqrt{n}$, then in such case ϵ_1 will tend to ∞ as n goes to ∞. This short discussion will be added to the manuscript for further clarity. • We remark that even if the ground truth is fair, it does not imply that the outcome will be fair as it is known in the fairness literature that the choice of model or algorithm has an effect in the final outcome. Thus, while the "relaxed" setting proposed by R3 is interesting and appealing as future work, we believe our work is a first step in the line of considering fairness constraints even in the scenario of having fair data. • Note that while square grids have $\Delta = 0$, this only implies that a single constraint is not sufficient to observe improvement in exact recovery. However, given that the multiplicity of the algebraic connectivity in square grids is 2, two constraints can help exact recovery, as shown in the figures above (please see bullet 3 for R2). • We thank R3 for his thorough review and feedback. We plan to incorporate the small suggestions to improve the paper presentation. Finally, regarding the KKT conditions, complementary slackness is an optimality condition that is fulfilled by $\boldsymbol{Y}$ and hence used to derive the sufficient condition on uniqueness.

R4: • As noted by R4, the constraint can have interpretations other than fairness and would be interesting, as future work, to study other fairness notions or interpretations. However, our results would still apply whenever one has linear equality constraints. • We disagree that the results are very close to that of non-fairness version. R1 and R3 note that we provide important connections between linear constraints and the eigenvalue gap of the Laplacian (Δ) along with the Fiedler vector $\boldsymbol{\pi}_2$, which to the best of our knowledge was unknown before. • Pure theoretical work has historically been welcomed to NeurIPS, which can be noted in the conference website. In addition, our work was submitted to the Statistical Learning Theory category as our main contribution is in the understanding of the effect of fairness constraints at inference time.