Author Response: Axioms for Learning from Pairwise Comparisons #11464

Review 1

Regarding the definition of Pareto optimality: Please refer to the explanation in lines 173–177, which is based on the idea of pooling different RUM-produced datasets. This context motivated our definition, and in this model the demanding restriction on a vs x pairs is appropriate. (Your example makes sense, and is reminiscent of PMC.)

Your question about lines 231–242: Note that we assume (line 238) that the number of comparisons is uniform across all pairs, so we will see many $\#_i$-comparisons of a vs x for every x. Since we assume that the difference $\beta_a - \beta_x$ has increased, we will (whp) see more comparisons that go $a > x$.

Theorems 5.3 and 6.3 for more than three alternatives: crucially, the proofs of these theorems construct universal counterexamples, and these can easily be extended to more than three alternatives. We will mention this.

"only 12 numbers": this is because we don't need counts for x vs x comparisons. We'll clarify. The claim of line 181 is indeed provable. Regarding ε-perturbations, we were using a continuous relaxation, but by choosing rational ε and blowing the counts up, we can achieve integral counts.

Review 2

On the existence of parameters leading to high probability of PMC violations: Our proof sketch in lines 181–183 suggests that if datasets are generated by a RUM, it satisfies the Pareto condition of Definition 3.1 for all pairs, and hence the dataset is very unlikely to lead to a PMC violation.

Regarding examples of other aggregation rules that satisfy PMC and separability: this is a good point. A simple example of such a rule is the one where for each alternative $x \in \mathcal{X}$, we let $\hat{\beta}_x$ be the number of comparisons where x wins minus the number where it loses. (In a sense, this is a generalization of Borda's rule.) This rule satisfies Pareto, monotonicity, and separability, but still fails PMC (due to Example 5.2). In fact, separability and PMC are incompatible since one can prove that all separable aggregators are variants of this counting rule. Overall, we don't think this rule is promising: for instance, on dataset $\#^1$ of Example 6.2, it places c above a, which seems wrong as we argue in lines 291–292. Thus, in the applications we have in mind, RUMs seem more appropriate than this simple counting scheme, also because RUMs will have more predictive accuracy.

Review 3

Regarding necessity of log-concavity assumptions (also raised in Review 4): indeed it would be desirable to know more about this. As Reviewer 4 notes, this is probably a difficult project, since uniqueness of the MLE is tricky to ascertain, and since the MLE cannot be computed anymore by straightforwardly solving a convex program.

Review 4

Thank you for pointing that we seem to have rediscovered conditions for existence and uniqueness, and that (for Bradley–Terry) these were already known by Ford (1957). Searching further, it turns out that Ford himself rediscovered the conditions, which were already contained in Zermelo (1928). It was separately pointed out to us that the conditions for general RUMs were stated in an ICML-18 paper by Zhao and Xia. In the revised version, we will be careful to reference the appropriate literature. Please note, though, that we did not view these results as contributions of the paper — that is why we deliberately put them in the model section. We just need them as lemmas for our main theorems about the axioms. For those proofs, we also need the more explicit (and apparently novel) bounds using the inf-norm.

Regarding combining the conditions about existence and uniqueness into one: in some places (chiefly for Pareto optimality), we only need existence. For Pareto optimality, as you note, we were not completely clear about existence requirements. We will say that the statement on line 180 of Definition 3.1 is conditioned on the existence of $\hat{\beta}$.

Regarding your proposed generalization of the Pareto definition: This is an interesting suggestion. It is possible that a condition like this can be satisfied if the MLE first normalizes counts so that all pairs are seen equally often. However, such a normalization might be a disadvantage on other examples.

Regarding whether monotonicity is a participation incentive: There are two separate strategic models at play here. In the context that you describe, the alternatives are strategic players (such as competitors in a sports tournament) and need to decide whether to engage in another game with an uncertain outcome. You are right that in this model, monotonicity does not necessarily give a participation incentive. However, we were motivated by a different model. We think of the problem as aggregating preferences or opinions from voters, and then a participation incentive must assure voters that if they report additional pairwise comparisons, then the aggregate will move to be more aligned with the voter's opinion. Monotonicity, as we define it, gives exactly this guarantee.

Regarding the uniqueness requirement in Definition 4.1: we include a uniqueness requirement here so it makes sense to compare the positions of alternatives on different datasets. In general, the definition makes sense applied to aggregators that are not always unique. To keep it general, we prefer not to refer to the function F in the definition of an axiom.