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# Supplementary Material: Achieving Rotational Invariance with Bessel-Convolutional Neural Networks

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## A Orthonormality of the Bessel basis

In this section, the mathematical proof for the orthonormality of the Bessel basis is provided. To do so, one needs two preliminary results.

### A.1 Lemma 1

$$\begin{aligned} \int_0^{2\pi} e^{i\theta(\nu' - \nu)} d\theta &= \int_0^{2\pi} \cos(\theta(\nu' - \nu)) + i \int_0^{2\pi} \sin(\theta(\nu' - \nu)) \\ &= 2\pi \delta_{\nu, \nu'}, \end{aligned} \quad (1)$$

since  $\nu' - \nu$  is always an integer.

### A.2 Lemma 2

We will need to use the Lommel's integrals, which are in our particular case

$$\begin{aligned} \int_0^R \rho J_\nu(k_{\nu,j}\rho) J_\nu(k_{\nu,j'}\rho) d\rho &= \\ \begin{cases} \frac{1}{k_{\nu,j'}^2 - k_{\nu,j}^2} [\rho (k_{\nu,j} J'_\nu(k_{\nu,j}\rho) J_\nu(k_{\nu,j'}\rho) - k_{\nu,j'} J'_\nu(k_{\nu,j'}\rho) J_\nu(k_{\nu,j}\rho))]_0^R & \text{if } k_{\nu,j} \neq k_{\nu,j'}, \\ \left[ \frac{\rho^2}{2} \left[ J_\nu'^2(k_{\nu,j}\rho) + \left(1 - \frac{\nu^2}{k_{\nu,j}^2 \rho^2}\right) J_\nu^2(k_{\nu,j}\rho) \right] \right]_0^R & \text{otherwise.} \end{cases} \end{aligned} \quad (3)$$

By evaluating the  $[\cdot]_0^R$ , and by taking into account that  $J'_\nu(k_{\nu,j}R) = 0$  (the condition we imposed to set the  $k_{\nu,j}$ ), these integrals lead to

$$\int_0^R \rho J_\nu(k_{\nu,j}\rho) J_\nu(k_{\nu,j'}\rho) d\rho = \begin{cases} 0 & \text{if } k_{\nu,j} \neq k_{\nu,j'} \\ \left( \frac{R^2}{2} - \frac{\nu^2}{2k_{\nu,j}^2} \right) J_\nu(k_{\nu,j}R) & \text{otherwise.} \end{cases} \quad (4)$$

$$= \left( \frac{R^2}{2} - \frac{\nu^2}{2k_{\nu,j}^2} \right) J_\nu(k_{\nu,j}R) \delta_{j,j'} \quad (5)$$

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\*B. Frénay and A. Mayer are co-last authors.

### A.3 Proof

$$\int_0^{2\pi} \int_0^R \rho [N_{\nu,j} J_\nu(k_{\nu,j}\rho) e^{i\nu\theta}]^* [N_{\nu',j'} J_{\nu'}(k_{\nu',j'}\rho) e^{i\nu'\theta}] d\theta d\rho \quad (6)$$

$$= \int_0^{2\pi} e^{i\theta(\nu'-\nu)} \int_0^R \rho N_{\nu,j} J_\nu(k_{\nu,j}\rho) N_{\nu',j'} J_{\nu'}(k_{\nu',j'}\rho) d\theta d\rho. \quad (7)$$

By using Lemma 1, it leads to

$$\int_0^{2\pi} \int_0^R \rho [N_{\nu,j} J_\nu(k_{\nu,j}\rho) e^{i\nu\theta}]^* [N_{\nu',j'} J_{\nu'}(k_{\nu',j'}\rho) e^{i\nu'\theta}] d\theta d\rho \quad (8)$$

$$= 2\pi \delta_{\nu,\nu'} \int_0^R \rho N_{\nu,j} J_\nu(k_{\nu,j}\rho) N_{\nu,j'} J_\nu(k_{\nu,j'}\rho) d\theta d\rho, \quad (9)$$

and by using Lemma 2, to

$$\int_0^{2\pi} \int_0^R \rho [N_{\nu,j} J_\nu(k_{\nu,j}\rho) e^{i\nu\theta}]^* [N_{\nu',j'} J_{\nu'}(k_{\nu',j'}\rho) e^{i\nu'\theta}] d\theta d\rho \quad (10)$$

$$= 2\pi N_{\nu,j}^2 \left( \frac{R^2}{2} - \frac{\nu^2}{2k_{\nu,j}^2} \right) J_\nu(k_{\nu,j}R) \delta_{\nu,\nu'} \delta_{j,j'}. \quad (11)$$

To conclude this proof, one can show that

$$N_{\nu,j}^2 = \frac{1}{2\pi \int_0^R \rho J_\nu^2(k_{\nu,j}\rho) d\rho} \quad (12)$$

$$= \frac{1}{2\pi \left( \frac{R^2}{2} - \frac{\nu^2}{2k_{\nu,j}^2} \right) J_\nu(k_{\nu,j}R)}, \quad (13)$$

and then finally,

$$\int_0^{2\pi} \int_0^R \rho [N_{\nu,j} J_\nu(k_{\nu,j}\rho) e^{i\nu\theta}]^* [N_{\nu',j'} J_{\nu'}(k_{\nu',j'}\rho) e^{i\nu'\theta}] d\theta d\rho = \delta_{\nu,\nu'} \delta_{j,j'}. \square \quad (14)$$

## B Checklist

1. For all authors...

- (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [\[Yes\]](#) The claim is that we achieve rigorous rotational invariance for  $\alpha \in [0, 2\pi)$ , which is theoretically proved in Section 4 and tested in Section 5.
- (b) Did you describe the limitations of your work? [\[Yes\]](#) In Section 4.4, we discuss the fact that using B-CNNs generally leads to an increase in the number of parameters in each layer. In Section 6, we point out that a deeper analysis needs to be performed to more easily chose the meta-parameters of B-CNNs.
- (c) Did you discuss any potential negative societal impacts of your work? [\[N/A\]](#)
- (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [\[Yes\]](#)

2. If you are including theoretical results...

- (a) Did you state the full set of assumptions of all theoretical results? [\[Yes\]](#)
- (b) Did you include complete proofs of all theoretical results? [\[Yes\]](#) Except for Equation (6), because it would be too long, but the proof is provided in the supplementary material.

3. If you ran experiments...

- (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [\[Yes\]](#) Our implementation of B-CNNs is provided in the supplemental material. All the instructions and codes to reproduce the results are provided, and all the data used are available following the references.
  - (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [\[Yes\]](#)
  - (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [\[Yes\]](#) In our experiments, 99% confidence intervals are provided (obtained over 40 independent runs). Standard deviations are also given for experiments in Section 5
  - (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [\[No\]](#) The training for all the architectures is performed using GPUs. However, the total amount of compute and the type of GPUs is not provided as our implementation of B-CNNs is not, for the moment, as well optimized as the two other methods we tested (G-CNNs and CNNs). The comparison would therefore not make much sense.
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
- (a) If your work uses existing assets, did you cite the creators? [\[Yes\]](#) For data, references are given, and for the use of Group-CNNs, authors are cited.
  - (b) Did you mention the license of the assets? [\[N/A\]](#)
  - (c) Did you include any new assets either in the supplemental material or as a URL? [\[Yes\]](#) Our B-CNNs implementation is provided in the supplemental material
  - (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [\[N/A\]](#)
  - (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [\[N/A\]](#)
5. If you used crowdsourcing or conducted research with human subjects...
- (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [\[N/A\]](#)
  - (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [\[N/A\]](#)
  - (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [\[N/A\]](#)