Evidently, this result implies that if we can determine a suitable bound for $\|e_i^\top A(I - QQ^\top)\|^2$ then we automatically get a proper bound for the element-wise approximations of Algorithm 1. If A has a fast decaying spectrum and Q captures the dominant eigenspace of A we can expect that our approximations are very accurate, even for small l. For the general case, however, the following Lemma 3 as well as the optimality of the JL lemma [31] already hint that this is not possible (see also Appendix II, Limitations of low-rank projections).

Lemma 3. *Let $A \in \mathbb{R}^{n\times d}$. For $1 \le k < d$, it holds that $\|e_i^\top (A - A_k)\|_2^2 \le \sigma_{k+1}^2(A) \le \frac{\|A_k\|_F^2}{k}$.*

Proof. Clearly, $\|e_i^\top (A - A_k)\|_2^2 \le \max_{\|x\|=1} \|x^\top (A - A_k)\|_2^2 = \sigma_{k+1}^2(A)$. For the second part we have that $\sigma_{k+1}^2(A) \le \frac{1}{k}\sum_{i=1}^k \sigma_i^2(A) = \frac{\|A_k\|_F^2}{k}$. □

2.1 Projecting rows on randomly chosen subspaces

To proceed further with the analysis, we show some length-preserving properties of the orthogonal projector $QQ^\top$, which is an orthogonal projector on a random subspace as obtained in line 3 of Algorithm 1. Note that Corollary 1 is stated for constant factor approximations. Here we provide a brief proof sketch. For the main result we refer to Lemma 8 in Appendix III.

Corollary 1 (Projection on $\mathrm{rowspace}(SA^\top A)$). *(Proof in the Appendix) Let $\delta \in (0, \frac{1}{2})$, $\bar{A}_k = A - A_k$, and S be such that*

(i) $S \sim \mathcal{D}$, where $\mathcal{D}$ is an $(1/3, \delta)$-OSE for any fixed k-dimensional subspace;

(ii) S is a $(1/3, \delta, n)$-JLT.

Fix JLT parameter $2n \to n$.

If Q is a matrix that forms an orthonormal basis for $\mathrm{rowspace}(SA^\top A)$, then, with probability at least $1 - 2\delta$, for all $i \in [n]$ simultaneously, it holds that

Fix constants, $1/2 \to 2$ and $3/2 \to 3$.

$$\|e_i^\top A(I - QQ^\top)\|^2 \le \|e_i^\top (\bar{A}_k)\|^2 + 2\frac{\sigma_{k+1}^2(A)}{\sigma_k^2(A)}\|e_i^\top A_k\|\|e_i^\top \bar{A}_k\| \le 3\|e_i^\top A\|\|e_i^\top \bar{A}_k\|.$$

Proof sketch. To prove the result it suffices to find a projector within $\mathrm{rowspace}(SA^\top A)$ with the desired properties. To do this, we consider the matrix $\Pi_k = V_k(SV_k\Sigma_k^2)^\dagger SA^\top A$, where V_k, Σ_k originate from the SVD of $A_k = U_k\Sigma_k V_k^\top$. Clearly, this Π_k is a rank-k matrix within $\mathrm{rowspace}(SA^\top A)$. After some algebra, the problem reduces to get a bound for the quantities

$$|e_i^\top AV_k(SV_k\Sigma_k^2)^\dagger S\bar{V}_k\bar{\Sigma}_k^2\bar{V}_k^\top A^\top e_i|,$$

for all $i \in [n]$. This is achieved by using Cauchy-Schwarz and by applying the OSE and JLT properties of S. □

Adapt proof-sketch to the corrected proof (see appendix).

Having all pieces in-place, we can finally bound the element-wise approximations of Algorithm 1.

Theorem 1. *(Proof in the Appendix) Let $A \in \mathbb{R}^{n\times d}$ and $n \ge d$. If we use Algorithm 1 with m matrix-vector queries to estimate the Euclidean lengths of the rows of A, then there exists a global constant C such that, as long as*

(i) $m \ge l \ge O(\log(n/\delta))$, such that G satisfies Lemma 1 and S forms an $(1/3, \delta, n)$-JLT,

(ii) $m \ge O(k + \log(1/\delta))$, such that S forms an $(1/3, \delta)$-OSE for a k-dimensional subspace,

Fix JLT parameter $2n \to n$.

then it holds that

$$\left|\tilde{x}_i - \|e_i^\top A\|^2\right| \le C\sqrt{\frac{\log(\frac{n}{\delta})}{l}}\|e_i^\top (A - A_k)\|\|e_i^\top A\| \le C\sqrt{\frac{\log(\frac{n}{\delta})}{lk}}\|A_k\|_F\|e_i^\top A\|,$$

for all $i \in [n]$ with probability at least $1 - 3\delta$.