A Finding an Initial Good Center

In this section we give, for completeness, the ρ-zCDP version of the algorithms for approximating P's optimal radius up to a constant factor and finding some θ_0 which is sufficiently close to the center of P's MEB. The algorithm itself is ridiculously simple, and has appeared before implicitly. We bring it here for two reasons: (a) completeness and (b) in its LDP-version, this algorithm's utility depends solely on $\sqrt{n}$. Thus, combining this algorithm with the Algorithm 5 of Section 5, we obtain a LDP-fPTAS for the MEB problem who's utility depends on $\sqrt{n}$ rather than the $n^{0.67}$-bound of [31] (at the expense of worse dependency on other parameters). This gives a clear improvement on previous algorithms for approximating the MEB problem when $n \to \infty$. Our algorithm requires a starting point θ_0 which is $R_{\max}$ away from all points in P (namely, $P \subset B(\theta_0, R_{\max})$, and a lower bound $r_{\min}$ on r_{opt}; and its overall utility bounds depends on $\log(R_{\max}/r_{\min})$. In a standard setting, where $P \subset [-B, B]^d$ and where all points lie on some grid $\mathcal{G}^d$ whose step-size is τ, we can set θ_0 as the origin and set $R_{\max} = B\sqrt{d}$ and $r_{\min} = \tau/2$, resulting in $O(\log(^{Bd}/_{\tau}))$-dependency. In the specific case where $r_{opt} = 0$ and all datapoints in P lie on the exact same grid point we can just return the closest grid point to the resulting θ once it get to a radius of $r = r_{min} = \tau/2$.

Algorithm 6 Noisy Average and Radius (GoodCenter)

Input: a set of n points P and parameters $\theta_0, R_{\max}$ and $r_{\min}$, such that $P \subset B(\theta_0, R_{\max})$ and $r_{opt} \geq r_{\min}$. Failure parameter $\beta \in (0, 1)$, privacy parameter ρ.

```
Set T ← ⌈log_2(R_max/r_min)⌉ + 1, X ← sqrt( 2T ln(4T/β) / ρ )
Set σ²_count ← T/ρ, σ²_sum ← T/ρ.
Init P^0 ← P, θ^0 ← θ_0, n_cur ← n and r_cur ← R_max.
for (t = 0, 1, 2, ..., T − 1) do
    P^t ← P^t ∩ B(θ^t, r_cur).
    Δ_sum ~ N(0, 4 r_cur² σ²_sum I_d)
    μ̃^t ← (Σ_{x∈P^t} x + Δ_sum)/n_cur
    Δ_count ← N(0, σ²_count)
    if (|P^t \ B(μ̃^t, ½ r_cur)| + Δ_count ≥ X) then return B(θ^t, r_cur)
    Update: r_cur ← ½ r_cur, n_cur ← n_cur − 2X, θ^{t+1} ← μ̃^t.
return B(θ^T, r_cur)
```

Theorem A.1. *Algorithm 6 is ρ-zCDP.*

Proof. The proof follows immediately from the fact that the L_2-global sensitivity of a count query is 1, and that the L_2-global sensitivity of a sum of datapoints in a ball of radius r_{cur} is at most $2r_{cur}$. The rest of the proof relies on the composition of $2T$ queries, each answered with a "budget" of $\frac{\rho}{2T}$-zCDP. ☐

Theorem A.2. *W.p. $\geq 1 - \beta$, given a set of points P of size n where $n \geq \max\{16T\sqrt{\frac{2T\ln(4T/\beta)}{\rho}}, 16\sqrt{\frac{T}{\rho}}(\sqrt{d} + \sqrt{2\ln(4T/\beta)})\}$, Algorithm 6 returns a ball $B(\theta^*, r^*)$ where (i) the set $P' = P \cap B(\theta^*, r^*)$ contains at least $n - \sqrt{\frac{8T^3\ln(4T/\beta)}{\rho}}$, and (ii) denoting $B(\theta(P'), r_{opt}(P'))$ as the MEB of P', we have that $r^* \leq 6r_{opt}$.*

Proof. Let $\mathcal{E}$ be the event where for any of the $\leq T$ draws of the Δ_{sum} and Δ_{count} it holds that

$$|\Delta_{count}| \leq \sqrt{\frac{2T\ln(4T/\beta)}{\rho}} \qquad \text{and} \qquad \|\Delta_{sum}\| \leq 2r_{cur}\sqrt{\frac{T}{\rho}}(\sqrt{d} + \sqrt{2\ln(4T/\beta)})$$

where again, standard union bound and Gaussian / χ^2-distribution concentration bounds give that $\Pr[\overline{\mathcal{E}}] \leq \beta$. So we continue the proof under the assumption that $\mathcal{E}$ holds.

In this case, in any iteration it must hold that $|P \setminus B(\mu^t, \frac{1}{2}r_{cur})| \leq 2X = \sqrt{\frac{8T\ln(4T/\beta)}{\rho}}$. It follows that all in all we remove in the process of Algorithm 6 at most $2XT$ points, and since $n \geq 16XT$

we have that in any iteration t it always holds that $n \geq |P^t| \geq n - 2Xt = n_{cur} \geq \frac{7n}{8} \geq 14XT$. Denoting in any iteration t the true mean of the points (remaining) in P^t as $\mu_t = \frac{1}{|P^t|}\sum_{x\in P^t} x$, and the center of the MED of P^t as θ_t, it follows that

$$\begin{aligned}
\|\tilde{\mu}^t - \mu^t\| = \|\tilde{\mu}^t - \theta_t - (\mu^t - \theta_t)\| &= \left\| \frac{\Delta_{sum} + \sum_{x\in P^t}(x-\theta_t)}{n_{cur}} - \frac{\sum_{x\in P^t}(x-\theta_t)}{|P^t|} \right\| \\
&\leq \left\| \frac{\Delta_{sum}}{n_{cur}} \right\| + \left\| \frac{\left(\sum_{x\in P^t}(x-\theta_t)\right)(|P^t| - n_{cur})}{|P^t| n_{cur}} \right\| \leq \frac{8\|\Delta_{sum}\|}{7n} + \|\mu^t - \theta_t\| \frac{2XT}{n_{cur}} \\
&\leq \frac{8 \cdot 2r_{cur}\sqrt{\frac{T}{\rho}}(\sqrt{d} + \sqrt{2\ln(4T/\beta)})}{7n} + \frac{r_{opt}(P^t)}{7} \leq \frac{r_{cur} + r_{opt}(P^t)}{7}
\end{aligned}$$

Since we assume $n \geq 16\sqrt{\frac{T}{\rho}}(\sqrt{d} + \sqrt{2\ln(4T/\beta)})$. Moreover, since $\|\mu^t - \theta_t\| \leq r_{opt}(P^t)$ it follows that $\|\tilde{\mu}^t - \theta_t\| \leq \frac{r_{cur} + 8r_{opt}(P^t)}{7}$. Now, as long as $r_{cur} \geq 6r_{opt}(P^t)$ we have that

$$\frac{r_{cur}}{2} \geq \frac{r_{cur}}{7} + \frac{5r_{cur}}{14} \geq \frac{r_{cur}}{7} + \frac{30r_{opt}(P^t)}{14} \geq r_{opt}(P^t) + \frac{r_{cur} + 8r_{opt}(P^t)}{7} \geq r_{opt}(P^t) + \|\tilde{\mu}^t - \theta_t\|$$

thus $B(\theta_t, r_{opt}(P^t)) \subset B(\tilde{\mu}^t, \frac{r_{cur}}{2})$ which implies that $|P^t \setminus B(\mu^t, \frac{1}{2}r_{cur})| = 0$, and so under $\mathcal{E}$ we continue to the next iteration.

And so, when we halt it must hold that r_{cur} (which is the r^* we return) must satisfy that $r_{cur} < 6r_{opt}(P^t)$. □

Corollary A.3. Algorithm 6 is a ρ-zCDP algorithm that, given n points on a grid $\mathcal{G} \subset [-B, B]^d$ of side-step τ where $n = \Omega(\sqrt{\frac{\log(Bd/\tau)}{\rho}}(\sqrt{d} + \sqrt{\log(Bd/\tau\beta)}))$ returns w.p. $\geq 1 - \beta$ a ball $B(\theta^*, r^*)$ where for $P' = P \setminus B(\theta^*, r^*)$ it holds that both $n - |P'| = O(\frac{\log(Bd/\tau)}{\sqrt{\rho}}\sqrt{\log(Bd/\tau\beta)}))$ and that w.r.t to $B(\theta_{opt}, r_{opt})$ which is the true MEB of P' we have that $\|\theta^* - \theta_{opt}\| \leq 6r_{opt}(P')$.

A.1 A Local-DP Version of Finding an Initial Good Center

Algorithm 7 LDP Noisy Average and Radius (LDP-GoodCenter)

Input: a set of n points P and some parameter $R_{\max}, \theta_0$ and $r_{\min}$, such that $P \subset B(\theta_0, R_{\max})$ and $r_{opt} \geq r_{\min}$. Failure parameter $\beta \in (0, 1)$, privacy parameter ρ.

```
Set T ← ⌈log_2(R_max/r_min)⌉ + 1, X ← sqrt(2nT ln(4T/β) / ρ)
σ²_count ← T/ρ, σ²_sum ← T/ρ.
Init θ^0 ← θ_0, and r_cur ← R_max.
for (t = 0, 1, 2, ..., T − 1) do
    Denote Π^t as the projection onto B(θ^t, r_cur).
    for each x ∈ P do
        Send Y_x ~ N(Π^t(x), 4 r²_cur σ²_sum I_d)
    μ̃^t ← (1/n) Σ_x Y_x
    for each x ∈ P do
        if (x ∉ B(μ̃^t, ½ r_cur)) then
            Send Z_x ~ N(1, σ²_count)
        else Send Z_x ~ N(0, σ²_count)
    if (Σ_x Z_x ≥ X) then return B(θ^t, r_cur)
    Update: r_cur ← ½ r_cur, θ^{t+1} ← μ̃^t.
return B(θ^T, r_cur)
```

Theorem A.4. *Algorithm 7 is a LDP algorithm in which each user maintains ρ-zCDP. Forthermore, w.p. $\geq 1 - \beta$, given a set of point P of size n where $n \geq \max\{16T\sqrt{\frac{2nT\ln(4T/\beta)}{\rho}}, 16\sqrt{\frac{nT}{\rho}}(\sqrt{d} +$*

$\sqrt{2\ln(4T/\beta)})\}$*, Algorithm 7 returns a ball* $B(\theta^*, r^*)$ *where the set* $P' = \{\Pi_{B(\theta^*,r^*)}(x) :\ x \in P\}$ *contains no more than* $2T\sqrt{\frac{2T\ln(4T/\beta)}{\rho}}$ *points for which* $x \neq \Pi_{B(\theta^*,r^*)}(x)$*; and denoting* $B(\theta(P'), r_{opt}(P'))$ *as the MEB of* P'*, it holds that* $\|\theta^* - \theta(P')\| \leq 8r*$*.*

The proof of Theorem A.4 is completely analogous to the proof of Theorems A.1 and A.2 using the fact that in each iteration t of the algorithm

$$\sum_x Y_x \sim \mathcal{N}\left(\sum_x \Pi^t(x),\ \ 4nr_{cur}^2\sigma_{sum}^2 I_d\right)$$
$$\sum_x Z_x \sim \mathcal{N}\left(|\{x\in P :\ x \notin B(\tilde{\mu}^t, r_{cur}/2)\}|,\ \ n\sigma_{count}^2\right)$$

Corollary A.5. Algorithm 7 is a ρ-zCDP algorithm in the local-model that, given n points on a grid $\mathcal{G} \subset [-B, B]^d$ of side-step τ where $n = \Omega(\frac{\log(Bd/\tau)}{\rho}(\sqrt{d} + \sqrt{\log(Bd/\tau\beta)})^2)$ returns w.p. $\geq 1-\beta$ a ball $B(\theta^*, r^*)$ where for the set $P' = \{\Pi_{B(\theta^*,r^*)}(x) :\ x \in P\}$ it holds that at most $O(\frac{\sqrt{n}\cdot\log(Bd/\tau)}{\sqrt{\rho}}\sqrt{\log(Bd/\tau\beta)})$ points are shifted in the projection (and the rest remain as they are in P) and that w.r.t to $B(\theta_{opt}, r_{opt})$ which is the true MEB of P' we have that $\|\theta^* - \theta_{opt}\| \leq 6r^*$.

Note that comparing Corollary A.5 with the approximation of [31], we have that they may omit $O(n^{0.67}\log(n/\tau))$-many points whereas we may omit only $\sqrt{n}\log^{3/2}(d/\tau)$ points. But, of course, they deal with a bounding ball for t points out of giving n, whereas we deal with the MEB problem.

B Using Noisy Mean

Here we continue the analysis detailed in Section 3.1. For completeness, we also bring the SQ-model version of the algorithm where in each iteration we obtain an approximated center $\tilde{\mu}^t$ where $\Delta^t = \tilde{\mu}^t_w - \mu^t_w$ is of magnitude propostional to γr. We modify Algorithm 2 so that our update scale shrinks by a constant factor to $\gamma^2/8$, namely we set $\theta^{t+1} \leftarrow (1-\frac{\gamma^2}{8})\theta^t + \frac{\gamma^2}{8}\tilde{\mu}^t_w$. We now prove that the revised algorithm still converges to a point close to θ_{opt}.

Lemma B.1. Applying Algorithm 2 with any $4r_{opt} \geq r \geq r_{opt}$ and any θ_0 where $\|\theta_0 - \theta_{opt}\| \leq 10r_{opt}$, where in each iteration we use an approximated mean $\tilde{\mu}^t_w = \mu^t_w + \Delta^t$ where $\|\Delta^t\| \leq \frac{\gamma r}{16} \leq \frac{\gamma r_{opt}}{4}$ we obtain a θ where $\|\theta - \theta_{opt}\| \leq \gamma r_{opt}$ in at most $16T = \frac{64}{\gamma^2}\ln(100/\gamma^2)$ iterations.

Proof. First, analogously to Lemma 3.2 we have that in each update step we get

$$\begin{aligned}\|\theta^{t+1} - \theta_{opt}\|^2 &= \left\|\left((1-\frac{\gamma^2}{8})\theta^t + \frac{\gamma^2}{8}\tilde{\mu}^t_w\right) - \theta_{opt}\right\|^2 = (1-\frac{\gamma^2}{8})^2\cdot\|\theta^t - \theta_{opt}\|^2 \\ &\quad + 2\frac{\gamma^2}{8}(1-\frac{\gamma^2}{8})\left(\langle\theta^t - \theta_{opt}, \mu^t_w - \theta_{opt}\rangle + \langle\theta^t - \theta_{opt}, \Delta^t\rangle\right) + (\frac{\gamma^2}{8})^2\cdot\|\mu^t_w - \theta_{opt} + \Delta^t\|^2 \\ &\leq (1-\frac{\gamma^2}{8})^2\cdot\|\theta^t - \theta_{opt}\|^2 + 2(\frac{\gamma^2}{8} - \frac{\gamma^4}{64})\cdot\left(\frac{1}{2}\|\theta^t - \theta_{opt}\|^2 + \|\theta^t - \theta_{opt}\|\cdot\frac{\gamma r_{opt}}{4}\right) \\ &\quad + (\frac{\gamma^2}{8})^2\cdot\left(2\|\mu^t_w - \theta_{opt}\|^2 + 2\frac{\gamma^2 r_{opt}^2}{4^2}\right) \\ &\leq (1-\frac{\gamma^2}{8})^2\|\theta^t - \theta_{opt}\|^2 + 2(\frac{\gamma^2}{8} - \frac{\gamma^4}{64})\cdot\|\theta^t - \theta_{opt}\|\left(\frac{1}{2}\|\theta^t - \theta_{opt}\| + \frac{\gamma r_{opt}}{4}\right) + \frac{3\gamma^4}{64}r_{opt}^2\end{aligned}$$

It follows that in each iteration where $\|\theta^t - \theta_{opt}\| \geq \gamma r_{opt}$ we get that

$$\begin{aligned}\|\theta^{t+1} - \theta_{opt}\|^2 &\leq (1 - \frac{2\gamma^2}{8} + \frac{\gamma^4}{64})\|\theta^t - \theta_{opt}\|^2 + 2(\frac{\gamma^2}{8} - \frac{\gamma^4}{64})\cdot\frac{3}{4}\|\theta - \theta_{opt}\|^2 + \frac{3\gamma^4 r_{opt}^2}{64} \\ &< (1-\frac{\gamma^2}{16})\|\theta^t - \theta_{opt}\|^2 + \frac{3\gamma^2}{64}\|\theta^t - \theta_{opt}\|^2 = (1-\frac{\gamma^2}{64})\|\theta^t - \theta_{opt}\|^2\end{aligned}$$

suggesting that after $16T = \frac{64}{\gamma^2}\ln(100/\gamma^2)$ iteration at most it must hold that

$$\|\theta^{16T} - \theta_{opt}\|^2 \leq \exp(-\frac{64}{\gamma^2}\ln(100/\gamma^2)\cdot\frac{\gamma^2}{64})\|\theta_0 - \theta_{opt}\|^2 \leq \frac{\gamma^2}{100}\cdot 100 r_{opt}^2 = \gamma^2 r_{opt}^2$$

As required. Similarly, if at some iteration t it holds that $\|\theta^t - \theta_{opt}\| < \gamma r_{opt}$ then we get that

$$\begin{aligned}\|\theta^{t+1} - \theta_{opt}\|^2 &\leq (1-\frac{\gamma^2}{8})^2\gamma^2 r_{opt}^2 + 2(\frac{\gamma^2}{8} - \frac{\gamma^4}{64})\cdot\frac{3}{4}\gamma^2 r_{opt}^2 + \frac{3\gamma^4 r_{opt^2}}{64}\\ &\leq \gamma^2 r_{opt}^2\left(1 - \frac{2\gamma^2}{8} + \frac{\gamma^4}{64} + \frac{3\gamma^2}{2\cdot 8} - \frac{3\gamma^4}{2\cdot 64} + \frac{3\gamma^2}{64}\right) \leq (1-\frac{\gamma^2}{64})\gamma^2 r_{opt}^2\end{aligned}$$

suggesting yet again that $\|\theta^\tau - \theta_{opt}\| < \gamma r_{opt}$ for all $\tau \geq t$. ☐

C Missing Proofs: DP Algorithm

C.1 Privacy Analysis

Lemma C.1. Algorithm 4 satisfies *ρ-zCDP*.

Proof. At each one of the RT iterations of the algorithm, we answer two queries to the input data: a counting query and a summation query. It is known that the L_2*-sensitivity* of a counting query is 1, therefore using the Gaussian mechanism theorem while setting $\sigma_{count}^2 = \frac{R(T+1)}{\rho}$ satisfies $\frac{\rho}{2R(T+1)}$-*zCDP*. Secondly, we know that all the points are bounded by a ball of radius $11r_0 \leq 44r_{opt} \leq 44r$ around θ_0, hence the summation query has L_2-sensitivity of $\leq 88r$. Thus, by setting $\sigma_{sum}^2 = \frac{RT(88r)^2}{\rho}$ we have that we answer each summation query using $\frac{\rho}{2T}$-zCDP. Due to sequential composition of zCDP [9], it holds that in all T iteration together we preserve $\left(\rho(1-\frac{1}{2R(T+1)})\right)$-zCDP. Lastly, we apply one last counting query which we answer using the Gaussian mechanism while satisfying $\frac{\rho}{2R(T+1)}$-zCDP, thus, overall we are ρ-zCDP. ☐

Corollary C.2. Algorithm 3 satisfies *ρ-zCDP*.

Proof. Since Algorithm 3 invokes $B = \lceil\log_2(\log_{1+\gamma}(4))\rceil$ calls to Algorithm 4 each preserving $\frac{\rho}{B}$-zCDP, Algorithm 3 is ρ-zCDP overall. ☐

C.2 Utility Analysis and Sample Complexity

Corollary C.3. [Corollary 4.2 restated] Given r_0 where $r_{opt} \leq r_0 \leq 4r_{opt}$ and a point θ_0 where $\|\theta_0 - \theta^*\| \leq 10r_{opt}$, w.p. $\geq 1-\beta$ Algorithm 3 is a $O(n\cdot\frac{\log^2(1/\gamma)\log(1/\beta)}{\gamma^2})$-time algorithm that returns a ball $B(\theta^*, r)$ where $r \leq (1+3\gamma)r_{opt}$ and where $|P \setminus B(\theta^*, r^*)| = O(\frac{\left(\sqrt{d}+\sqrt{\log(\log(1/\beta)/\gamma)}\right)\sqrt{\log(1/\gamma)\log(1/\beta)}}{\gamma\sqrt{\rho}})$.

Proof. The result follows directly from the fact that Algorithm 3 invokes $B = O(\log(1/\gamma))$ calls to Algorithm 4, with a privacy budget of $O(\rho/\log(1/\gamma))$ each and with a failure probability of $O(\beta/\log(1/\gamma))$ each. Plugging those into the bound of Lemma 4.1 together with the fact that $T = O(\gamma^{-2}\log(1/\gamma))$ yields the resulting bound. Note that, denoting the "correct" $i^* = \min\{i \geq 0 : \frac{r_0}{4}(1+\gamma)^i \geq r_{opt}\}$, under the event that no invocation of Algorithm 4 fails, each time we execute the binary search with a value of $i_{cur} \geq i^*$ we obtain some $\theta_{cur} \neq \perp$. Due to the nature of the binary search and the fact that upon finding $\theta_{cur} \neq \perp$ we set $i_{\max} = i_{cur}$, it must follows that we return a ball of radius $(1+\gamma)r^* = (1+\gamma)\cdot\frac{r_0}{4}\cdot(1+\gamma)^i$ for some $i \leq i^*$, and so $r^* \leq (1+\gamma)^2 r_{opt} \leq (1+3\gamma)r_{opt}$. Lastly, the runtime of Algorithm 4 is $O(nRT)$ making the runtime of Algorithm 3 to be $O(nRTB) = O(\frac{n\log^2(1/\gamma)\log(1/\beta)}{\gamma^2})$ as required. ☐

C.3 Application: Subsample Stable Functions

Much like the work of [21], our work too is applicable as a DP-aggregator in a Subsample-and-Aggregate [30] framework. We say that a point $p \in \mathbb{R}^d$ is (r,β)-*stable* for some function $f : \mathcal{X}^* \to \mathbb{R}^d$ if there exists $m(r,\beta)$ such that for any input $S \subset \mathcal{X}^n$ a random subsample of m entries of S input datapoints returns w.p. $\geq 1-\beta$ a value close to p, namely, $\Pr_{S' \subset S, |S|=m}[\|c - f(S')\| \leq r] \geq 1-\beta$.

Theorem C.4. *Fix $\rho, \gamma, \beta > 0$. There exists some constant $C > 0$ such that the following holds. Suppose $f : \mathcal{X}^* \to \mathbb{R}^d$ is a function that has a (r,β)-stable point. Then, there exists a ρ-zCDP algorithm that takes an input a dataset $S \subset \mathcal{X}^n$ and w.p.$\geq 1-\beta$ returns a $((1+\gamma)r, \beta/2k)$-stable point provided that $n \geq k \cdot m(r, \beta/2k)$ for $k = \frac{C\left(\sqrt{d}+\sqrt{\log(\log(1/\beta)/\gamma)}\right)\sqrt{\log(1/\gamma)\log(1/\beta)}}{\gamma\sqrt{\rho}}$. Furthermore, if finding $f(S')$ for any S' containing $m(r,\beta/2k)$-many datapoint takes T time, then our algorithm runs in time $O(k\mathsf{T} + k \cdot \frac{\log^2(1/\gamma)\log(1/\beta)}{\gamma^2})$.*

Proof. The proof simply partitions the n inputs points of S into k disjoint and random subsets $S'_1, S'_2, ..., S'_k$. W.p. $\geq 1-\beta/2$ it holds that $\|f(S'_i) - c\| \leq r$ for every subset S'_i, and then we apply our $(1+\gamma)$ approximation over this dataset of k many points (with a failure probability of $\beta/2$) and returns the resulting center-point. ☐

This results improves on Theorem 18 of [21] in both the runtime and the required number of subsamples, at the expense of requiring *all* subsamples to be close to the point p rather than just many of the points.

D Missing Proofs: Local-DP Algorithm

Claim D.1. Algorithm 5 is a local-model ρ-zCDP.

Proof. The proof is very similar to the proof of Lemma C.1 — where we apply basically the same accounting, noticing that each $x \in P$ is in charge of randomizing her own data, making this algorithm LDP. ☐

Lemma D.2. W.p. $\geq 1-\beta$, applying Algorithm 4 with $r \geq r_{opt}$ and an initial center θ_0 s.t. $\|\theta_0 - \theta_{opt}\| \leq 10r_{opt}$ returns a point θ^t where $|P \setminus B(\theta^t, (1+\gamma)r)| \leq 88\sqrt{\frac{nRT}{\rho}}\left(\sqrt{d} + \sqrt{2\ln(4nRT/\beta_0)}\right) + \sqrt{\frac{2R(T+1)\log(4R(T+1)/\beta_0)}{\rho}}$.

Proof. Analogously to the proof of Lemma 4.1, we use the similar definitions: in each iteration t we denote n^t_w as the true number of datapoints in P outside the ball $n^t_w = |\{x \in P:\ x \notin B(\theta^t, r)\}|$,[8] μ^t_w as their true mean $\mu^t_W = \frac{1}{n^t_w}\sum_{x \notin B(\theta^t,r)} x$, and v^t_w as the difference of the true mean and the current center $v^t_w = \mu^t_w - \theta^t = \frac{1}{n^t_w}\sum_{x \notin B(\theta^t,r)}(x - \theta^t)$. We thus define the events

$$\mathcal{E}_1 := \text{in all } T+1 \text{ iterations, } |\tilde{n}^t_w - n^t_w| \leq \sqrt{\frac{2nR(T+1)\log(4(T+1)/\beta)}{\rho}}$$

$$\mathcal{E}_2 := \text{in all } T \text{ iterations, } \|\sum_x Z^t_x - n^t_w v^t_w\| \leq \frac{88r\sqrt{nRT}}{\sqrt{\rho}}\left(\sqrt{d} + \sqrt{2\ln(4T/\beta)}\right)$$

Proving that both $\Pr[\overline{\mathcal{E}}_1] \leq \beta/2$ and $\Pr[\overline{\mathcal{E}}_2] \leq \beta/2$ is rather straight-forward. In each iteration t it holds that $\sum_x Y^t_x \sim \mathcal{N}(n^t_w, n\sigma^2_{count})$ as the sum on n independent Gaussians, and so we merely apply standard Gaussian concentration bounds together with the union bound over all $T+1$ iterations. Similarly, in each iteration t it holds that $\sum_x Z^t_x \sim \mathcal{N}(n^t_w(\mu^t_x - \theta^t), n\sigma^2_{sum} I_d)$. So standard bounds on the concentration of the χ^2_d-distribution assert that the L_2-distance between the random draw from such a d-dimensional Gaussian and its mean is $> \sqrt{n\sigma^2_{sum}}(\sqrt{d} + \sqrt{2\ln(4T/\beta)}$ w.p. $< \frac{\beta}{2T}$, after which we apply the union-bound on all T iterations. We continue the rest of the proof conditioning on both $\mathcal{E}_1$ and $\mathcal{E}_2$ holding.

[8] Where technically, in the last steps of the algorithm, $n^T_w = |\{x \in P:\ x \notin B(\theta^T, (1+\gamma)r)\}|$.

Again, due to our if-condition, we make an update-step only when $\tilde{n}^t_w$ is large, which, under $\mathcal{E}_1$ implies that

$$n^t_w \geq \frac{88\sqrt{nRT}}{\sqrt{\rho}}\left(\sqrt{d}+\sqrt{2\ln(4RT/\beta_0)}\right) - \sqrt{\frac{2nR(T+1)\ln(4R(T+1)/\beta_0)}{\rho}} \geq 44|\Delta_{count}|$$

and then proving that the distribution which we use to make an update-step satisfies the conditions detailed in (2) w.h.p. is precisely the same proof (using the independence of Δ_{count} and Δ_{sum} and the fact that $\mathbb{E}[\Delta_{sum}] = 0$).

Invoking Corollary 3.5 we have that if we make all T updates then indeed $\|\theta^T - \theta_0\| \leq \gamma r$ and so $|P \setminus B(\theta^T, (1+\gamma)R)| = 0$. So under $\mathcal{E}_1$ Algorithm 4 returns θ^T. Otherwise, at some iteration we do not make an update step, which under $\mathcal{E}_1$ suggest that

$$n^t_w = |P \setminus B(\theta^t, r)| \leq 88\sqrt{\frac{nRT}{\rho}}\left(\sqrt{d}+\sqrt{2\ln(4nRT/\beta_0)}\right) + \sqrt{\frac{2R(T+1)\log(4R(T+1)/\beta_0)}{\rho}}$$

□

Corollary D.3. Algorithm 3 altered so it invokes $B = O(\log(1/\gamma))$ calls to Algorithm 5 (instead of Algorithm 4) is a $O(\frac{\log(1/\beta)\log^2(1/\gamma)}{\gamma^2})$-rounds ρ-zCDP algorithm in the local-model that takes $O(n \cdot \frac{\log^2(1/\gamma)\log(1/\beta)}{\gamma^2})$-time; and that returns a ball $B(\theta^*, r^*)$ such that $r^* \leq (1+3\gamma)r_{opt}$ and $|P \setminus B(\theta^*, r^*)| = O(\frac{\sqrt{n}\left(\sqrt{d}+\sqrt{\log(\log(1/\beta)/\gamma)}\right)\sqrt{\log(1/\gamma)\log(1/\beta)}}{\gamma\sqrt{\rho}})$.

Proof. The proof follows from using the bound of Lemma D.2 with $T = O(\gamma^{-2}\log(1/\gamma))$, and with a privacy budget of ρ/B and failure probability of β/B in each invocation of Algorithm 5. □

E Experiments

In this section we give an experimental evaluation of our algorithm on three synthetic datasets and one real dataset. We emphasize that our experiment should be perceived merely as a proof-of-concept experiment aimed at the possibility of improving the algorithm's analysis, and not a thorough experimentation for a ready-to-deploy code. We briefly explain the experimental setup below.

Goal. We set to investigate the performance of our algorithm, and seeing whether the performance is similar across different types of input and across a range of parameters. In addition, we wondered whether in practice our algorithm halts prior to concluding all $T = O(\gamma^{-2}\ln(1/\gamma))$ iterations.

Experiment details. We conducted experiments solely with Algorithm 4 with update-step that uses a constant learning rate of $\gamma^2/8$, feeding it the true r_{opt} of each given dataset as its r parameter. By default, we used the following set of parameters. Our domain in the synthetic experiments is $[-5, 5]^{10}$ (namely, we work in the 10-dimensional space), and our starting point θ_0 is the origin. The default values of our privacy parameter is $\rho = 0.3$, of the approximation constant is 1.2 (namely $\gamma = 0.2$), and of the failure probability is $\beta = e^{-9} \approx 0.00012$. We set the maximal number of repetitions T just as detailed in Algorithm 4, which depends on γ.

We varied two of the input parameters, ρ and γ, and also the data-type. We ran experiments with $\rho \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and with $\gamma \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$. Based on the values of ρ and γ we computed $n_0 = \frac{\sqrt{RT}(\sqrt{d}+\sqrt{\ln(4RT/\beta_0)})}{\sqrt{\rho}}$ which we used as our halting parameter. In all experiments involving a synthetic dataset, we set the input size n to be $n = 640n_0$.

We varied also the input type, using 3 synthetically generated datasets and one real-life dataset:

- Spherical Gaussian: we generated samples from a d-dimensional Gaussian $\mathcal{N}(v, I_d)$, where $v \in \mathbb{R}^d$ is a random shift vector. We discarded each point that did not fall in $[-5, 5]^{10}$.
- Product Distribution: we generated samples from a d-dimensional Bernoulli distribution with support $\{-1, 1\}^d$ with various probabilities for each dimension — where for each

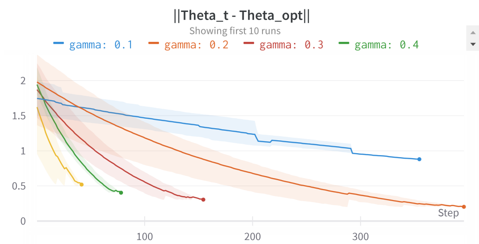
coordinate $i \in [10]$ we set $\Pr[x_i = 1] = 2^{-i}$. This creates a "skewed" distribution whose mean is quite far from its 1-center. In order for the 1-center not to coincide with $\theta_0 = \bar{0}$ we shifted this cube randomly in the grid.

- Conditional Gaussian: we repeated the experiment with the spherical Gaussian only this time we conditioned our random draws so that no coordinate lies in the $[0, 0.5]$-interval. This skews the mean of the distribution to be < 0 in each coordinate, but leaves the 1-center unaltered. Again, we shifted the Gaussian to a random point $v \in [-5, 5]^d$.
- "Bar Crawl: Detecting Heavy Drinking": a dataset taken from the freely available UCI Machine Learning Repository [1] which collected accelerometer data from participants in a college bar crawl [26]. We truncated the data to only its 3 x-, y- and z-coordinates, and dropped any entry outside of $[-1, 1]^3$, and since it has two points $(-1, -1, -1)$ and $(1, 1, 1)$ then its 1-center is the origin (so we shifted the data randomly in the $[-5, 5]^3$ cube). This left us with $n = 12,921,593$ points. Note that the data is taken from a very few participants, so our algorithm gives an event-level privacy [17].

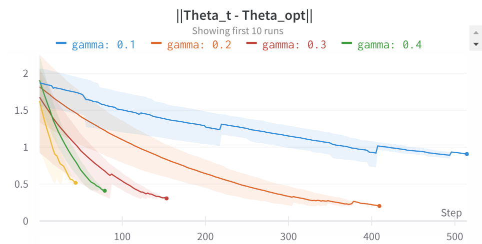
We ran our experiments in Python, on a (fairly standard) Intel Core i7 2.80 GHz with 16GB RAM and they run in time that ranged from 15 seconds (for $\gamma = 0.5$) to 2 hours (for $\gamma = 0.1$).

Results. The results are given in Figures 2, 3, where we plotted the distance of θ^t to θ_{opt} for each set of parameters across $t = 10$ repetitions. As evident, we converged to a good approximation of the MEB in all settings. We halt the experiment (i) if $\|\theta_t - \theta_{opt}\| \leq \gamma r_{opt}$, or (ii) if there are not enough wrong points, or (iii) if $t > 2500$ indicating that the run isn't converging. Indeed, the number of iterations until convergence does increase as γ decreases; but, rather surprisingly, varying ρ has a small effect on the halting time. This is somewhat expected as T has no dependency on ρ whereas its dependency on γ is proportional to γ^{-2}, but it is evident that as ρ increases our mean-estimation in each iteration becomes more accurate, so one would expect a faster convergence. Also unexpectedly, our results show that even for datasets whose mean and 1-center aren't close to one another (such as the Conditional Gaussian or Product-Distribution), the number of iterations until convergence remains roughly the same (see for example Figure 2 vs. 3).

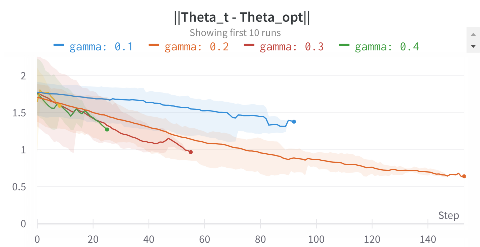
Conclusions. Our experiments suggest that indeed our bound T is a worst-case bound, where in all experiments we concluded in about $7 - 50$ times faster than the bound of Algorithm 4. This suggests that perhaps one would be better off if instead of partitioning the privacy budget equally across all T iterations, they devise some sort of adaptive privacy budgeting. (E.g., using $3\rho/4$ budget on the first $T/4$ iterations and then the remaining $\rho/4$ budget on the latter $3T/4$ iterations.) Such adaptive budgeting is simple when using zCDP, as it does not require "privacy odometers" [33].



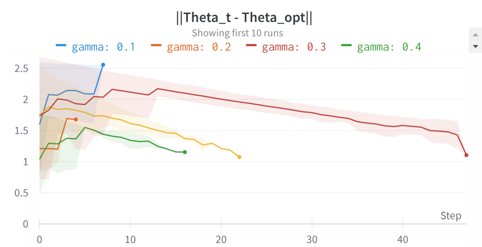
(a) Spherical Gaussian



(b) Conditional Gaussian

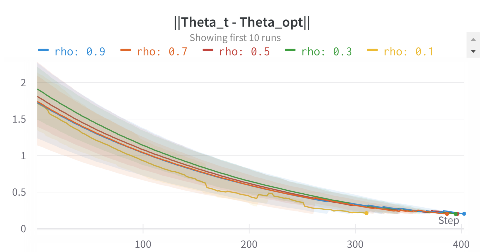


(c) Product Distribution

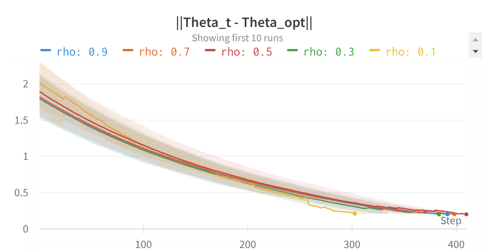


(d) Bar Crawl: Detecting Heavy Drinking

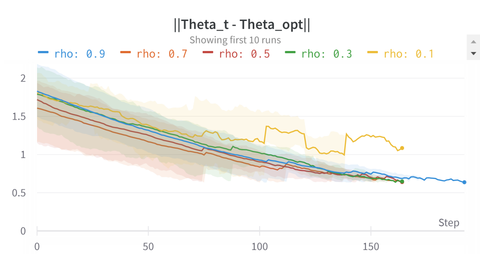
Figure 2: The distance of θ^t to θ_{opt} as a function of t – the iteration number, for $\rho = 0.3$ and $\gamma \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$. Each curve corresponds to a different γ value. In all experiments the number of iterations until convergence does increase as γ decreases, except for $\gamma = 0.1$ where it halts because there were not enough wrong points. Note that for $\gamma = 0.1$ for Bar Crawl dataset (figure 2d) we didn't converge due to its size.



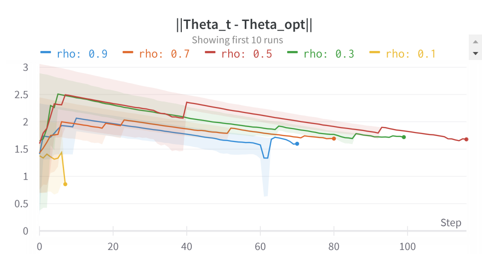
(a) Spherical Gaussian



(b) Conditional Gaussian



(c) Product Distribution



(d) Bar Crawl: Detecting Heavy Drinking

Figure 3: The distance of θ^t to θ_{opt} as a function of t – the iteration number, for $\gamma = 0.2$ and $\rho \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Each curve corresponds to a different ρ value. In all experiments varying ρ has a small effect on the halting time.