A Appendix

A.1 Proofs

In this section we restate and provide proofs of the statements made in the main text.

Property 1. *For any* $\Pi \subseteq \mathbb{\Pi}$, $\mathcal{V} \subseteq \mathbb{V}$ *and* $\mathcal{M} \subseteq \mathbb{M}$, *it follows that* $\mathcal{M}^k(\Pi, \mathcal{V}; 0) = \mathcal{M}^k(\Pi, \mathcal{V})$ *and* $\mathcal{M}^\infty(\Pi; 0) = \mathcal{M}^\infty(\Pi)$.

Proof. Any $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; 0)$ satisfies $\|\tilde{\mathcal{T}}_\pi^k v - \mathcal{T}_\pi^k v\| = 0 \; \forall \pi \in \Pi, \forall v \in \mathcal{V}$. Similarly any $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V})$ satisfies analogous equality constraints $\tilde{\mathcal{T}}_\pi^k v = \mathcal{T}_\pi^k v \; \forall \pi \in \Pi, \forall v \in \mathcal{V}$. Since $\|\cdot\|$ is a norm, we know that $\|\tilde{\mathcal{T}}_\pi^k v - \mathcal{T}_\pi^k v\| = 0 \iff \tilde{\mathcal{T}}_\pi^k v = \mathcal{T}_\pi^k v$, hence $\mathcal{M}^k(\Pi, \mathcal{V}; 0) = \mathcal{M}^k(\Pi, \mathcal{V})$. The same logic applies to APVE classes. □

Property 2. *For any* $\epsilon \in \bar{\mathbb{R}}^+$, $\mathcal{M} \subseteq \bar{\mathcal{M}} \subseteq \mathbb{M}$, $\Pi \subseteq \Pi' \subseteq \mathbb{\Pi}$ *and* $\mathcal{V} \subseteq \mathcal{V}' \subseteq \mathbb{V}$, *it follows that*

$$\mathcal{M}^k(\Pi', \mathcal{V}'; \epsilon) \subseteq \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \bar{\mathcal{M}}^k(\Pi, \mathcal{V}; \epsilon). \tag{7}$$

Proof. An AVE class, $\mathcal{M}^k(\Pi', \mathcal{V}'; \epsilon)$ satisfies a series of constraints of the form $\|\tilde{\mathcal{T}}_\pi^k v - \mathcal{T}_\pi^k v\| \leq \epsilon$ for each pair of $\pi, v \in \Pi' \times \mathcal{V}'$. Considering another pair of sub-sets $\Pi \subseteq \Pi'$ and $\mathcal{V} \subseteq \mathcal{V}'$, we can partition the first pair as follows:

$$\Pi' \times \mathcal{V}' = (\Pi' \setminus \Pi \times \mathcal{V}' \setminus \mathcal{V}) \uplus (\Pi' \setminus \Pi \times \mathcal{V}) \uplus (\Pi \times \mathcal{V}' \setminus \mathcal{V}) \uplus (\Pi \times \mathcal{V})$$

accordingly,

$$\begin{aligned}\mathcal{M}^k(\Pi', \mathcal{V}'; \epsilon) &= \mathcal{M}^k(\Pi' \setminus \Pi, \mathcal{V}' \setminus \mathcal{V}; \epsilon) \cap \mathcal{M}^k(\Pi' \setminus \Pi, \mathcal{V}; \epsilon) \cap \mathcal{M}^k(\Pi, \mathcal{V}' \setminus \mathcal{V}; \epsilon) \cap \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \\ &\subseteq \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon),\end{aligned}$$

satisfying the first subset relation in Eq. 7. For the next subset relation, we simply note that

$$\bar{\mathcal{M}}^k(\Pi, \mathcal{V}; \epsilon) = (\bar{\mathcal{M}} \setminus \mathcal{M})^k(\Pi, \mathcal{V}; \epsilon) \cup \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \supseteq \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon),$$

completing the proof. □

Property 3. *For any* $\Pi \subseteq \mathbb{\Pi}$, $\mathcal{V} \subseteq \mathbb{V}$ *and* $\epsilon, \epsilon' \in \bar{\mathbb{R}}^+$ *such that* $\epsilon' \geq \epsilon$, *it follows that*

$$\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon'). \tag{8}$$

Proof. For any $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$ a number of AVE constraints are respected: $\|\tilde{\mathcal{T}}_\pi^k v - \mathcal{T}_\pi^k v\| \leq \epsilon$ for each pair $\pi, v \in \Pi \times \mathcal{V}$. Since $\epsilon' \geq \epsilon$, it follows that $\|\tilde{\mathcal{T}}_\pi^k v - \mathcal{T}_\pi^k v\| \leq \epsilon \leq \epsilon'$ as well and hence $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon')$. Thus $\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon')$ as needed. □

Proposition 1. *For any* $\epsilon \in \bar{\mathbb{R}}^+$, $\Pi, \Pi' \subseteq \mathbb{\Pi}$, $\mathcal{V}, \mathcal{V}' \subseteq \mathbb{V}$ *and* $k, K \in \mathbb{Z}^+$ *there exists some* $\epsilon' \in \bar{\mathbb{R}}^+$ *such that*

$$\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^K(\Pi', \mathcal{V}'; \epsilon'). \tag{9}$$

Moreover, if $\mathcal{M}$, $\mathcal{V}$ *and* $\mathcal{V}'$ *are bounded then* ϵ' *is finite.*

Proof. Denote $v_{\text{max}} = \max_{s \in \mathcal{S}, v \in \mathcal{V} \cup \mathcal{V}'} v(s)$, $\tilde{r}_{\text{max}} = \max_{s \in \mathcal{S}, a \in \mathcal{A}, \tilde{m} \in \mathcal{M}} r(s, a)$ and consider any $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$. We can then write

$$\|\tilde{\mathcal{T}}_\pi^K v - \mathcal{T}_\pi^K v\| \leq \max_s |\tilde{\mathcal{T}}_\pi^K v(s)| + \max_s |\mathcal{T}_\pi^K v(s)| \leq 2 \max\{\tilde{r}_{\text{max}}, r_{\text{max}}\} \tfrac{1-\gamma^K}{1-\gamma} + \gamma^K v_{\text{max}}$$

for any $\pi \in \Pi'$, $v \in \mathcal{V}'$ and $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$.

Clearly, when $\epsilon' = \infty$ the desired subset relation holds, as $\mathcal{M}^K(\Pi', \mathcal{V}'; \infty) = \mathcal{M} \supseteq \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$ for any choices of sets, orders and ϵ. Additionally, when $\mathcal{M}$, $\mathcal{V}$ and $\mathcal{V}'$ are bounded, we know that $\tilde{r}_{\text{max}}$ and v_{max} are finite. Thus, by selecting a finite $\epsilon' > 2 \max\{\tilde{r}_{\text{max}}, r_{\text{max}}\} \frac{1-\gamma^K}{1-\gamma} + \gamma^K v_{\text{max}}$, we obtain $\tilde{m} \in \mathcal{M}^K(\Pi', \mathcal{V}'; \epsilon')$ and thus $\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^K(\Pi', \mathcal{V}'; \epsilon')$ as needed.

□

Proposition 2. *For any $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$ it follows that*

$$\|v_{\tilde{\pi}_*} - v_*\| \leq 2 \cdot \mathcal{E}_\epsilon(\Pi, \mathcal{V}, k \,|\, \mathbb{\Pi}, \infty),$$

where $\tilde{\pi}_*$ is any optimal policy of $\tilde{m}$.

Proof. From Proposition 1, we know that a minimum tolerated error, $\epsilon' = \mathcal{E}_\epsilon(\Pi, \mathcal{V}, k \,|\, \mathbb{\Pi}, \infty)$, exists such that $\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^\infty(\mathbb{\Pi}; \epsilon')$. We can then consider the performance of models in $\mathcal{M}^\infty(\mathbb{\Pi}; \epsilon')$. For any $\tilde{m} \in \mathcal{M}^\infty(\mathbb{\Pi}; \epsilon')$ we can write:

$$\begin{aligned} 0 &\geq \tilde{v}_{\pi_*}(s) - \tilde{v}_{\tilde{\pi}_*}(s) \\ &= (\tilde{v}_{\pi_*}(s) - v_{\pi_*}(s)) + (v_{\pi_*} - v_{\tilde{\pi}_*}(s)) + (v_{\tilde{\pi}_*}(s) - \tilde{v}_{\tilde{\pi}_*}(s)) \end{aligned} \tag{19}$$

for any $s \in \mathcal{S}$ where π_* and $\tilde{\pi}_*$ are arbitrary optimal policies in the environment and $\tilde{m}$ respectively and $\tilde{v}_\pi$ denotes the model's value of a policy π.

Since $\tilde{m} \in \mathcal{M}^\infty(\mathbb{\Pi}; \epsilon')$ we know the first and third terms are bounded below by $-\epsilon'$, giving:

$$\begin{aligned} &0 \geq v_*(s) - v_{\tilde{\pi}_*}(s) - 2\epsilon' \\ \Longrightarrow\; &2\epsilon' \geq v_*(s) - v_{\tilde{\pi}_*}(s) \geq 0 \\ \Longrightarrow\; &\|v_* - v_{\tilde{\pi}_*}\| \leq 2\epsilon', \end{aligned} \tag{20}$$

as needed. □

Proposition 3. *For any $\epsilon \in \bar{\mathbb{R}}^+$, $\Pi \subseteq \mathbb{\Pi}$, $\mathcal{V} \subseteq \mathbb{V}$ such that $v \in \mathcal{V} \implies \mathcal{T}_\pi v \in \mathcal{V}\ \forall \pi \in \Pi$ and $k, K \in \mathbb{Z}^+$ such that k divides K, we have that*

$$\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^K(\Pi, \mathcal{V}; \tfrac{\epsilon \cdot (1-\gamma^K)}{1-\gamma^k}). \tag{10}$$

Proof. Let $K = nk$, and consider a model $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$. It follows for any $\pi \in \Pi$ and $v \in \mathcal{V}$ that

$$\begin{aligned} \|\tilde{\mathcal{T}}_\pi^K v - \mathcal{T}_\pi^K v\| &= \|\tilde{\mathcal{T}}_\pi^k \tilde{\mathcal{T}}_\pi^{K-k} v - \mathcal{T}_\pi^k \mathcal{T}_\pi^{K-k} v\| \\ &= \|\tilde{\mathcal{T}}_\pi^k \tilde{\mathcal{T}}_\pi^{K-k} v - \mathcal{T}_\pi^k \mathcal{T}_\pi^{K-k} v + \tilde{\mathcal{T}}_\pi^k \mathcal{T}_\pi^{K-k} v - \tilde{\mathcal{T}}_\pi^k \mathcal{T}_\pi^{K-k} v\| \\ &\leq \|\tilde{\mathcal{T}}_\pi^k \mathcal{T}_\pi^{K-k} v - \mathcal{T}_\pi^k \mathcal{T}_\pi^{K-k} v\| + \|\tilde{\mathcal{T}}_\pi^k \tilde{\mathcal{T}}_\pi^{K-k} v - \tilde{\mathcal{T}}_\pi^k \mathcal{T}_\pi^{K-k} v\| \\ &\overset{(1)}{\leq} \epsilon + \|\tilde{\mathcal{T}}_\pi^k \tilde{\mathcal{T}}_\pi^{K-k} v - \tilde{\mathcal{T}}_\pi^k \mathcal{T}_\pi^{K-k} v\| \\ &\overset{(2)}{\leq} \epsilon + \gamma^k \|\tilde{\mathcal{T}}_\pi^{K-k} v - \mathcal{T}_\pi^{K-k} v\| \end{aligned} \tag{21}$$

where (1) follows from the assumption on $\mathcal{V}$ and (2) follows from the fact that $\tilde{\mathcal{T}}_\pi$ is a contraction. Next, using induction we can say that:

$$\begin{aligned} \|\tilde{\mathcal{T}}_\pi^K v_\pi - \mathcal{T}_\pi^K v_\pi\| &\leq \epsilon \cdot \left(1 + \gamma^k + \gamma^{2k} + \cdots + \gamma^{(n-1)k}\right) \\ &= \epsilon \cdot \sum_{t=0}^{n-1} \gamma^{kt} \\ &= \epsilon \cdot \tfrac{1-(\gamma^k)^n}{1-\gamma^k} \\ &= \epsilon \cdot \tfrac{1-\gamma^K}{1-\gamma^k} \end{aligned} \tag{22}$$

where the last equality follows because $K = nk$.

This suffices to show that $\tilde{m} \in \mathcal{M}^K(\Pi, \mathcal{V}; \epsilon \cdot \frac{1-\gamma^K}{1-\gamma^k})$ and thus: $\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^K(\Pi, \mathcal{V}; \epsilon \cdot \frac{1-\gamma^K}{1-\gamma^k})$ as needed. □

Corollary 1. *For any set of policies $\Pi \subseteq \mathbb{\Pi}$, set of functions $\mathcal{V} \in \mathbb{V}$ such that $\{v_\pi : \pi \in \Pi\} \subseteq \mathcal{V}$ and $k \in \mathbb{Z}^+$, it follows that*

$$\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^\infty(\Pi; \tfrac{\epsilon}{1-\gamma^k}). \tag{11}$$

Proof.

$$\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) = \bigcap_{\pi \in \Pi} \bigcap_{v \in \mathcal{V}} \mathcal{M}^k(\{\pi\}, \{v\}; \epsilon) \subseteq \bigcap_{\pi \in \Pi} \mathcal{M}^k(\{\pi\}, \{v_\pi\}; \epsilon) \tag{23}$$

where the subset-relation holds from our assumption that $\{v_\pi : \pi \in \Pi\} \subseteq \mathcal{V}$.

Next we examine $\tilde{m} \in \mathcal{M}^k(\{\pi\}, \{v_\pi\}; \epsilon)$ for individual $\pi \in \Pi$. We know that for any such model:

$$\begin{aligned} \|\tilde{\mathcal{T}}_\pi^{nk} v_\pi - v_\pi\| &\leq \|\tilde{\mathcal{T}}_\pi^{nk} v_\pi - \tilde{\mathcal{T}}_\pi^{k} v_\pi\| + \|\tilde{\mathcal{T}}_\pi^{k} v_\pi - v_\pi\| \\ &\leq \gamma^k \|\tilde{\mathcal{T}}_\pi^{(n-1)k} v_\pi - v_\pi\| + \epsilon. \end{aligned}$$

By repeatedly applying this inequality we can obtain:

$$\|\tilde{\mathcal{T}}_\pi^{nk} v_\pi - v_\pi\| \leq \sum_{t=0}^{n-1} \epsilon \cdot \gamma^{(tk)} = \epsilon \cdot \tfrac{1-\gamma^{nk}}{1-\gamma^k}.$$

Next, from the continuity of $\|\cdot\|$, we can take limits to obtain:

$$\epsilon \cdot \tfrac{1}{1-\gamma^k} \geq \lim_{n \to \infty} \|\tilde{\mathcal{T}}_\pi^{nk} v_\pi - v_\pi\| = \| \lim_{n \to \infty} \tilde{\mathcal{T}}_\pi^{nk} v_\pi - v_\pi\| = \|\tilde{v}_\pi - v_\pi\|,$$

giving us that $\mathcal{M}^k(\{\pi\}, \{v_\pi\}; \epsilon) \subseteq \mathcal{M}^\infty(\{\pi\}; \epsilon \cdot \frac{1}{1-\gamma^k})$. We can plug this result back into Eq. 23 to obtain:

$$\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \bigcap_{\pi \in \Pi} \mathcal{M}^k(\{\pi\}, \{v_\pi\}; \epsilon) \subseteq \bigcap_{\pi \in \Pi} \mathcal{M}^\infty(\{\pi\}; \epsilon \cdot \tfrac{1}{1-\gamma^k}) = \mathcal{M}^\infty(\Pi; \epsilon \cdot \tfrac{1}{1-\gamma^k}),$$

as needed. □

Proposition 4. *For any set of policies $\Pi \subseteq \mathbb{\Pi}$, set of functions $\mathcal{V} \in \mathbb{V}$, $c > 1$ and error $\epsilon \in \bar{\mathbb{R}}^+$, we have*

$$\mathcal{M}^k(\Pi, c\text{-vspan}(\mathcal{V}); \epsilon) \subseteq \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^k(\Pi, c\text{-vspan}(\mathcal{V}); c \cdot \epsilon). \tag{13}$$

Proof. Clearly, $\mathcal{V} \subseteq c\text{-vspan}(\mathcal{V})$ and thus $\mathcal{M}^k(\Pi, c\text{-vspan}(\mathcal{V}); \epsilon) \subseteq \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$. We now prove that $\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^k(\Pi, \mathcal{V}; c \cdot \epsilon)$. We first consider any $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$ and $v' \in c\text{-vspan}(\mathcal{V})$. Since $v' \in c\text{-vspan}(\mathcal{V})$ we can write $v' = \sum_{i=1}^n \alpha_i v_i$ where $v_i \in \mathcal{V}$ for each i and $\sum_{i=1}^n |\alpha_i| \leq c$. From here we observe:

$$\begin{aligned} \|\tilde{\mathcal{T}}_\pi^k v' - \mathcal{T}_\pi^k v'\| &= \|\tilde{\mathcal{T}}_\pi^k (\sum_{i=1}^n \alpha_i v_i) - \mathcal{T}_\pi^k (\sum_{i=1}^n \alpha_i v_i)\| \\ &\leq \| \sum_{i=1}^n \alpha_i (\tilde{\mathcal{T}}_\pi^k v_i - \mathcal{T}_\pi^k v_i)\| \\ &\leq \sum_{i=1}^n |\alpha_i| \|\tilde{\mathcal{T}}_\pi^k v_i - \mathcal{T}_\pi^k v_i\| \\ &\leq \sum_{i=1}^n |\alpha_i| \epsilon \\ &\leq c \cdot \epsilon \end{aligned} \tag{24}$$

which shows that $\mathcal{M}^k(\Pi, c\text{-vspan}(\mathcal{V}); c \cdot \epsilon)$ as needed. □

Corollary 2. *When either $c = 1$ or $\epsilon = 0$, for any $\Pi \subseteq \mathbb{\Pi}$, $\mathcal{V} \subseteq \mathbb{V}$ it follows that*

$$\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) = \mathcal{M}^k(\Pi, c\text{-vspan}(\mathcal{V}); \epsilon). \tag{14}$$

Proof. The proof follows directly from Proposition 4. When either $c \in \{0, 1\}$ the left-most and right-most terms in Eq. 13 are equal, squeezing $\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) = \mathcal{M}^k(\Pi, c\text{-vspan}(\mathcal{V}); \epsilon)$ as needed. □

Proposition 5.

1. ***(Asymmetry)*** *For any* $\mathcal{V} \subseteq \mathcal{V}' \subseteq \mathcal{V}'' \subseteq \mathbb{V}$ *it follows that*

$$0 = \beta(\mathcal{V}||\mathcal{V}) \leq \beta(\mathcal{V}||\mathcal{V}') \leq \beta(\mathcal{V}||\mathcal{V}'') \quad \textit{and} \quad 0 = \beta(\mathcal{V}''||\mathcal{V}'') \leq \beta(\mathcal{V}'||\mathcal{V}'') \leq \beta(\mathcal{V}||\mathcal{V}'').$$

2. ***(Convex, Compact*** $\mathcal{V}$***)*** *When* $\mathcal{V}$ *is convex and compact it follows that*

$$\beta(\mathcal{V}||\mathcal{V}') = \beta(\mathcal{V}||\text{1-vspan}(\mathcal{V}')).$$

Proof.

1. Recall $\beta(\mathcal{V}||\mathcal{V}') = \max_{v' \in \mathcal{V}'} \min_{v \in \mathcal{V}} \|v' - v\|$. Increasing the size of $\mathcal{V}'$ means that more elements can be maximized over, thereby increasing $\beta(\mathcal{V}||\mathcal{V}')$. Similarly, increasing the size of $\mathcal{V}$ means that more elements can be minimized over, thereby decreasing $\beta(\mathcal{V}||\mathcal{V}')$. When $\mathcal{V} = \mathcal{V}'$, we know that

$$0 \leq \beta(\mathcal{V}||\mathcal{V}') = \max_{v' \in \mathcal{V}'} \min_{v \in \mathcal{V}} \|v' - v\| \leq \max_{v' \in \mathcal{V}'} \|v' - v'\| = 0,$$

where second inequality follows since $\mathcal{V} = \mathcal{V}'$.

2. We begin by considering the function $g(v') = \min_{v \in \mathcal{V}} \|v - v'\|$. We begin by showing that this function is convex. Consider $v_1', v_2' \in \mathcal{V}'$ and denote $v_1 = \text{argmin}_{v \in \mathcal{V}} \|v - v_1'\|$ and $v_2 = \text{argmin}_{v \in \mathcal{V}} \|v - v_2'\|$. Then for any $\lambda \in [0, 1]$ we can write:

$$\begin{aligned} \lambda g(v_1') + (1 - \lambda) g(v_2') &= \lambda \|v_1' - v_1\| + (1 - \lambda) \|v_2' - v_2\| \\ &\geq \|(\lambda v_1' + (1 - \lambda) v_2') - (\lambda v_1 + (1 - \lambda) v_2)\| \end{aligned} \tag{25}$$

since $\mathcal{V}$ is convex $(\lambda v_1 + (1 - \lambda) v_2) \in \mathcal{V}$, thus:

$$\begin{aligned} \|(\lambda v_1' + (1 - \lambda) v_2') - (\lambda v_1 + (1 - \lambda) v_2)\| &\geq \min_{v \in \mathcal{V}} \|(\lambda v_1' + (1 - \lambda) v_2') - v\| \\ &= g(\lambda v_1' + (1 - \lambda) v_2') \end{aligned} \tag{26}$$

which suffices to show that g is a convex function.

Next we consider any element $v' \in \text{1-vspan}(\mathcal{V}')$ such that $v' = \sum_i \alpha_i v_i'$ with $\sum_i \alpha_i = 1$ and $\alpha_i \geq 0$ for all i. We can then write:

$$g(v') = g(\sum_i \alpha_i v_i') \leq \sum_i \alpha_i g(v_i') \leq \max_i g(v_i') \leq \max_{v' \in \mathcal{V}'} g(v') = \beta(\mathcal{V}||\mathcal{V}') \tag{27}$$

Since $g(v') \leq \beta(\mathcal{V}||\mathcal{V}')$ for every $v' \in \text{1-vspan}(V')$ it then follows that

$$\beta(\mathcal{V}||\text{1-vspan}(\mathcal{V}')) = \max_{v' \in \text{1-vspan}(\mathcal{V}')} g(v') \leq \beta(\mathcal{V}||\mathcal{V}'). \tag{28}$$

We obtain the reverse equality by noting that $\mathcal{V}' \subseteq \text{1-vspan}(\mathcal{V}')$ and thus $\beta(\mathcal{V}||\mathcal{V}') \leq \beta(\mathcal{V}||\text{1-vspan}(\mathcal{V}'))$. Hence $\beta(\mathcal{V}||\text{1-vspan}(\mathcal{V}')) = \beta(\mathcal{V}||\mathcal{V}')$ as needed.

□

Proposition 6. *For any* $\Pi \in \mathbb{\Pi}$, $\mathcal{V}, \mathcal{V}' \in \mathbb{V}$ *and* $\epsilon \in \bar{\mathbb{R}}^+$, *it follows that*

$$\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^k(\Pi, \mathcal{V}'; \epsilon + 2\gamma^k \beta(\mathcal{V}||\mathcal{V}')),$$

moreover, if $\mathcal{V}$ *is convex and compact, we obtain:*

$$\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^k(\Pi, \text{1-vspan}(\mathcal{V}'); \epsilon + 2\gamma^k \beta(\mathcal{V}||\mathcal{V}')).$$

Proof. Fix an arbitrary model $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$ and any $\pi \in \Pi$. We now select some $v' \in \mathcal{V}'$ and examine the tolerance with which $\tilde{m}$ is value equivalent with respect to $\{\pi\}$ and $\{v'\}$.

Notice that for *any* $v \in \mathcal{V}$ we can write

$$\begin{aligned}
\|\tilde{\mathcal{T}}_\pi^k v' - \mathcal{T}_\pi^k v'\| &= \|\tilde{\mathcal{T}}_\pi^k v' - \mathcal{T}_\pi^k v' + \tilde{\mathcal{T}}_\pi^k v - \tilde{\mathcal{T}}_\pi^k v\| \\
&\leq \|\tilde{\mathcal{T}}_\pi^k v' - \tilde{\mathcal{T}}_\pi^k v\| + \|\tilde{\mathcal{T}}_\pi^k v - \mathcal{T}_\pi^k v'\| \\
&= \|\tilde{\mathcal{T}}_\pi^k v' - \tilde{\mathcal{T}}_\pi^k v\| + \|\tilde{\mathcal{T}}_\pi^k v - \mathcal{T}_\pi^k v' + \mathcal{T}_\pi^k v - \mathcal{T}_\pi^k v\| \\
&\leq \|\tilde{\mathcal{T}}_\pi^k v' - \tilde{\mathcal{T}}_\pi^k v\| + \|\tilde{\mathcal{T}}_\pi^k v - \mathcal{T}_\pi^k v\| + \|\mathcal{T}_\pi^k v - \mathcal{T}_\pi^k v'\| \qquad (29) \\
&\overset{(1)}{\leq} 2\gamma^k \|v' - v\| + \|\tilde{\mathcal{T}}_\pi^k v - \mathcal{T}_\pi^k v\| \\
&\overset{(2)}{\leq} 2\gamma^k \|v' - v\| + \epsilon
\end{aligned}$$

where (1) follows from the Bellman operators $\tilde{\mathcal{T}}_\pi$ and $\mathcal{T}_\pi$ being contractions and (2) follows the assumption that $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V})$.

Since the above upper bound on $\|\tilde{\mathcal{T}}_\pi^k v' - \mathcal{T}_\pi^k v'\|$ holds for any $v \in \mathcal{V}$ we can write that

$$\|\tilde{\mathcal{T}}_\pi^k v' - \mathcal{T}_\pi^k v'\| \leq \epsilon + 2\gamma^k \min_{v \in \mathcal{V}} \|v' - v\|. \qquad (30)$$

Thus far we have shown that $\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^k(\Pi, \{v'\}; \epsilon + 2\gamma^k \min_{v \in \mathcal{V}} \|v' - v\|)$. To find a tolerance that holds for all $v' \in \mathcal{V}'$ we simply take a maximum over the element-wise tolerance:

$$\max_{v' \in \mathcal{V}} \epsilon + 2\gamma^k \min_{v \in \mathcal{V}} \|v' - v\| = \epsilon + 2\gamma^k \beta(\mathcal{V}||\mathcal{V}') \qquad (31)$$

This completes the proof. □

Theorem 2. *For any $\tilde{m} \in \mathcal{M}^k(\Pi, \mathcal{V}; \epsilon)$ it follows that*

$$\|v_* - v_{\tilde{\pi}_*}\| \leq \tfrac{2}{1-\gamma^k} \cdot \min_{c \geq 1} \left(c \cdot \epsilon + 2\gamma^k \beta(c\text{-vspan}(\mathcal{V})||\mathbb{V}_{\Pi})\right), \qquad (17)$$

where $\tilde{\pi}_*$ is an optimal policy of $\tilde{m}$.

Proof. From Theorem 1, we know by tolerating an error of

$$\epsilon' = \tfrac{1}{1-\gamma^k} \min_{c \geq 1} (c \cdot \epsilon + 2\gamma^k \beta(c\text{-vspan}\mathcal{V})||\mathbb{V}_{\Pi})),$$

that $\mathcal{M}^k(\Pi, \mathcal{V}; \epsilon) \subseteq \mathcal{M}^\infty(\Pi; \epsilon')$. Thus $\mathcal{E}_\epsilon(\Pi, \mathcal{V}, k \,|\, \Pi, \infty) \leq \epsilon'$. By applying Proposition 2, we obtain $\|v_* - v_{\tilde{\pi}_*}\| \leq 2\epsilon'$ as needed. □

Corollary 3. *Let $\hat{\mathbb{V}}_{\Pi} = \{\hat{v}_\pi : \pi \in \Pi\}$ be a set of approximate value functions satisfying $\|v_\pi - \hat{v}_\pi\| \leq \epsilon_{approx}$ for all $\pi \in \Pi$. Then for any $\tilde{m} \in \mathcal{M}^k(\Pi, \hat{\mathbb{V}}_{\Pi}; \epsilon)$ it follows that:*

$$\|v_* - v_{\tilde{\pi}_*}\| \leq \frac{2(\epsilon + 2\gamma^k \epsilon_{approx})}{1 - \gamma^k},$$

where $\tilde{\pi}_*$ is any optimal policy in $\tilde{m}$.

Proof. From the definition of $\hat{\mathbb{V}}_{\Pi}$, we know that $\beta(\mathbb{V}_{\Pi}||\hat{\mathbb{V}}_{\Pi}) \leq \epsilon_{\text{approx}}$. Thus by Proposition 6 and Corollary 1 we know that

$$\mathcal{M}^k(\Pi, \hat{\mathbb{V}}_{\Pi}; \epsilon) \subseteq \mathcal{M}^k(\Pi, \mathbb{V}_{\Pi}; \epsilon + 2\gamma^k \epsilon_{\text{approx}}) \subseteq \mathcal{M}^\infty(\Pi; \frac{\epsilon + 2\gamma^k \epsilon_{\text{approx}}}{1 - \gamma^k}),$$

thus $\mathcal{E}_\epsilon(\Pi, \hat{\mathbb{V}}_{\Pi}, k \,|\, \Pi, \infty) \leq \frac{\epsilon + 2\gamma^k \epsilon_{\text{approx}}}{1-\gamma^k}$, which gives us the desired performance bound by an application of Theorem 2. □