A Mathematical Preliminaries

In this section, we recall some basic facts about measure and probability theory that we need for the development in the main body of the paper. We follow Çınlar [11].

A.1 Measure Theory

Suppose that E is a set. We first define the notion of a σ-algebra. A non-empty collection $\mathcal{E}$ of E is called a *σ-algebra* on E if it is closed under complements and countable unions, that is, if

(i) $A \in \mathcal{E} \implies E \backslash A \in \mathcal{E}$;

(ii) $A_1, A_2, ... \in \mathcal{E} \implies \cup_{n=1}^{\infty} A_n \in \mathcal{E}$

[11, p.2]. We call $\{\emptyset, E\}$ the *trivial σ-algebra* of E. If $\mathcal{C}$ is an arbitrary collection of subsets of E, then the smallest σ-algebra that contains $\mathcal{C}$, or equivalently, the intersection of all σ-algebras that contain $\mathcal{C}$, is called the *σ-algebra generated by* $\mathcal{C}$, and is denoted $\sigma\mathcal{C}$.

A *measurable space* is a pair $(E, \mathcal{E})$, where E is a set and $\mathcal{E}$ is a σ-algebra on E [11, p.4].

Suppose $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces. For $A \in \mathcal{E}$ and $B \in \mathcal{F}$, we define the *measurable rectangle* $A \times B$ as the set of all pairs (x, y) with $x \in A$ and $y \in B$. We define the *product σ-algebra* $\mathcal{E} \otimes \mathcal{F}$ on $E \times F$ as the σ-algebra generated by the collection of all measurable rectangles. The measurable space $(E \times F, \mathcal{E} \otimes \mathcal{F})$ is the *product* of $(E, \mathcal{E})$ and $(F, \mathcal{F})$ [11, p.4]. More generally, if $(E_1, \mathcal{E}_1), ..., (E_n, \mathcal{E}_n)$ are measurable spaces, their product is

$$\bigotimes_{i=1}^{n}(E_i, \mathcal{E}_i) = (\bigtimes_{i=1}^{n} E_i, \bigotimes_{i=1}^{n} \mathcal{E}_i),$$

where $E_1 \times ... \times E_n$ is the set of all n-tuples $(x_1, ..., x_n)$ with x_i in E_i for $i = 1, ..., n$ and $\mathcal{E}_1 \otimes ... \otimes \mathcal{E}_n$ is the σ-algebra generated by the *measurable rectangles* $A_1 \times ... \times A_n$ with A_i in $\mathcal{E}_i$ for $i = 1, ..., n$ [11, p.44]. If T is an arbitrary (countable or uncountable) index set and $(E_t, \mathcal{E}_t)$ is a measurable space for each $t \in T$, the *product space* of $\{E_t : t \in T\}$ is the set $\bigtimes_{t \in T} E_t$ of all collections $(x_t)_{t \in T}$ with $x_t \in E_t$ for each $t \in T$. A rectangle in $\bigtimes_{t \in T} E_t$ is a subset of the form

$$\bigtimes_{t \in T} A_t = \{x = (x_t)_{t \in T} \in \bigtimes_{t \in T} E_t : x_t \in A_t \text{ for each } t \text{ in } T\}$$

where A_t differs from E_t for only a finite number of t. It is said to be measurable if $A_t \in \mathcal{E}_t$ for every t (for which A_t differs from E_t). The σ-algebra on $\bigtimes_{t \in T} E_t$ generated by the collection of all measurable rectangles is called the *product σ-algebra* and is denoted by $\bigotimes_{t \in T} \mathcal{E}_t$ [11, p.45].

A collection $\mathcal{C}$ of subsets of E is called a p-system if it is closed under intersections [11, p.2]. If two measures μ and ν on a measurable space $(E, \mathcal{E})$ with $\mu(E) = \nu(E) < \infty$ agree on a p-system generating $\mathcal{E}$, then μ and ν are identical [11, p.16, Proposition 3.7].

Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. A mapping $f : E \to F$ is *measurable* if $f^{-1}B \in \mathcal{E}$ for every $B \in \mathcal{F}$ [11, p.6].

Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Let f be a bijection between E and F, and let $\hat{f}$ denote its functional inverse. Then, f is an *isomorphism* if f is measurable relative to $\mathcal{E}$ and $\mathcal{F}$, and $\hat{f}$ is measurable with respect to $\mathcal{F}$ and $\mathcal{E}$. The measurable spaces $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are *isomorphic* if there exists an isomorphism between them [11, p.11].

A measurable space $(E, \mathcal{E})$ is a *standard measurable space* if it is isomorphic to $(F, \mathcal{B}_F)$ for some Borel subset F of $\mathbb{R}$. Polish spaces with their Borel σ-algebra are standard measurable spaces [11, p.11].

Let $A \subset E$. Its *indicator*, denoted by 1_A, is the function defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

[11, p.8]. Obviously, 1_A is $\mathcal{E}$-measurable if and only if $A \in \mathcal{E}$. A function $f : E \to \mathbb{R}$ is said to be *simple* if it is of the form

$$f = \sum_{i=1}^{n} a_i 1_{A_i}$$

for some $n \in \mathbb{N}$, $a_1, ..., a_n \in \mathbb{R}$ and $A_1, ..., A_n \in \mathcal{E}$ [11, p.8]. The $A_1, ..., A_n \in \mathcal{E}$ can be chosen to be a measurable partition of E, and is then called the *canonical form* of the simple function f. A positive function on E is $\mathcal{E}$-measurable if and only if it is the limit of an increasing sequence of positive simple functions [11, p.10, Theorem 2.17].

A *measure* on a measurable space $(E, \mathcal{E})$ is a mapping $\mu : \mathcal{E} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$;

(ii) $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for every disjoint sequence (A_n) in $\mathcal{E}$

[11, p.14]. A *measure space* is a triplet $(E, \mathcal{E}, \mu)$, where $(E, \mathcal{E})$ is a measurable space and μ is a measure on it.

A measurable set B is said to be *negligible* if $\mu(B) = 0$, and an arbitrary subset of E is said to be *negligible* if it is contained in a measurable negligible set. The measure space is said to be *complete* if every negligible set is measurable [11, p.17].

Next, we review the notion of integration of a real-valued function $f : E \to \mathbb{R}$ with respect to μ [11, p.20, Definition 4.3].

(a) Let $f : E \to [0, \infty]$ be simple. If its canonical form is $f = \sum_{i=1}^{n} a_i 1_{A_i}$ with $a_i \in \mathbb{R}$, then we define

$$\int f d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

(b) Suppose $f : E \to [0, \infty]$ is measurable. Then by above, we have a sequence (f_n) of positive simple functions such that $f_n \to f$ pointwise. Then we define

$$\int f d\mu = \lim_{n \to \infty} \int f_n d\mu,$$

where $\int f_n d\mu$ is defined for each n by (a).

(c) Suppose $f : E \to [-\infty, \infty]$ is measurable. Then $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$ are measurable and positive, so we can define $\int f^+ d\mu$ and $\int f^- d\mu$ as in (b). Then we define

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

provided that at least one term on the right be positive. Otherwise, $\int f d\mu$ is undefined. If $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$, then we say that f is *integrable*.

Finally, we review the notion of *transition kernels*, which are crucial in the consideration of conditional distributions. Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Let K be a mapping $E \times \mathcal{F} \to [0, \infty]$. Then, K is called a *transition kernel* from $(E, \mathcal{E})$ into $(F, \mathcal{F})$ if

(a) the mapping $x \mapsto K(x, B)$ is measurable for every set $B \in \mathcal{F}$; and

(b) the mapping $B \mapsto K(x, B)$ is a measure on $(F, \mathcal{F})$ for every $x \in E$.

A transition kernel from $(E, \mathcal{E})$ into $(F, \mathcal{F})$ is called a *probability transition kernel* if $K(x, F) = 1$ for all $x \in E$. A probability transition kernel K from $(E, \mathcal{E})$ into $(E, \mathcal{E})$ is called a *Markov kernel on* $(E, \mathcal{E})$ [11, p.37,39,40].

A.2 Probability Theory

Now we translate the above measure-theoretic notions into the language of probability theory, and introduce some additional concepts. A *probability space* is a measure space $(\Omega, \mathcal{H}, \mathbb{P})$ such that

$\mathbb{P}(\Omega) = 1$ [11, p.49]. We call Ω the *sample space*, and each element $\omega \in \Omega$ an *outcome*. We call $\mathcal{H}$ a collection of *events*, and for any $A \in \mathcal{H}$, we read $\mathbb{P}(A)$ as the *probability that the event A occurs* [11, p.50].

A *random variable* taking values in a measurable space $(E, \mathcal{E})$ is a function $X : \Omega \to E$, measurable with respect to $\mathcal{H}$ and $\mathcal{E}$. The *distribution* of X is the measure μ on $(E, \mathcal{E})$ defined by $\mu(A) = \mathbb{P}(X^{-1}A)$ [11, p.51]. For an arbitrary set T, let X_t be a random variable taking values in $(E, \mathcal{E})$ for each $t \in T$. Then the collection $\{X_t : t \in T\}$ is called a *stochastic process* with *state space* $(E, \mathcal{E})$ and *parameter set* T [11, p.53].

Henceforth, random variables are defined on $(\Omega, \mathcal{H}, \mathbb{P})$ and take values in $[-\infty, \infty]$. We define the *expectation* of a random variable $X : \Omega \to [-\infty, \infty]$ as $\mathbb{E}[X] = \int_\Omega X d\mathbb{P}$ [11, p.57-58]. We also define the *conditional expectation* [11, p.140, Definition 1.3]. Suppose $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{H}$.

(a) Suppose X is a positive random variable. Then the *conditional expectation of X given $\mathcal{F}$* is any positive random variable $\mathbb{E}_\mathcal{F} X$ satisfying

$$\mathbb{E}[VX] = \mathbb{E}\left[V \mathbb{E}_\mathcal{F} X\right]$$

for all $V : \Omega \to [0, \infty]$ measurable with respect to $\mathcal{F}$.

(b) Suppose $X : \Omega \to [-\infty, \infty]$ is a random variable. If $\mathbb{E}[X]$ exists, then we define

$$\mathbb{E}_\mathcal{F} X = \mathbb{E}_\mathcal{F} X^+ - \mathbb{E}_\mathcal{F} X^-,$$

where $\mathbb{E}_\mathcal{F} X^+$ and $\mathbb{E}_\mathcal{F} X^-$ are defined in (a).

Next, we define *conditional probabilities*, and regular versions thereof [11, pp.149-151]. Suppose $H \in \mathcal{H}$, and let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{H}$. Then the *conditional probability* of H given $\mathcal{F}$ is defined as

$$\mathbb{P}_\mathcal{F} H = \mathbb{E}_\mathcal{F} 1_H.$$

Let $Q(H)$ be a version of $\mathbb{P}_\mathcal{F} H$ for every $H \in \mathcal{H}$. Then $Q : (\omega, H) \mapsto Q_\omega(H)$ is said to be a *regular version* of the conditional probability $\mathbb{P}_\mathcal{F}$ provided that Q be a probability transition kernel from $(\Omega, \mathcal{F})$ into $(\Omega, \mathcal{H})$. Regular versions exist if $(\Omega, \mathcal{H})$ is a standard measurable space [11, p.151, Theorem 2.7].

The *conditional distribution* of a random variable X given $\mathcal{F}$ is any transition probability kernel $L : (\omega, B) \mapsto L_\omega(B)$ from $(\Omega, \mathcal{F})$ into $(E, \mathcal{E})$ such that

$$P_\mathcal{F}\{Y \in B\} = L(B) \qquad \text{for all } B \in \mathcal{E}.$$

If $(E, \mathcal{E})$ is a standard measurable space, then a version of the conditional distribution of X given $\mathcal{F}$ exists [11, p.151].

Suppose that T is a totally ordered set, i.e. whenever $r, s, t \in T$ with $r < s$ and $s < t$, we have $r < t$ and for any $s, t \in T$, exactly one of $s < t$, $s = t$ and $t < s$ holds [15, p.62]. For each $t \in T$, let $\mathcal{F}_t$ be a sub-σ-algebra of $\mathcal{H}$. The family $\mathcal{F} = \{\mathcal{F}_t : t \in T\}$ is called a *filtration* provided that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$ [11, p.79]. A *filtered probability space* $(\Omega, \mathcal{H}, \mathcal{F}, \mathbb{P})$ is a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ endowed with a filtration $\mathcal{F}$.

Finally, we review the notion of *independence* and *conditional independence*. For a fixed integer $n \geq 2$, let $\mathcal{F}_1, ..., \mathcal{F}_n$ be sub-σ-algebras of $\mathcal{H}$. Then $\{\mathcal{F}_1, ..., \mathcal{F}_n\}$ is called an *independency* if

$$\mathbb{P}\left(H_1 \cap ... \cap H_n\right) = \mathbb{P}\left(H_1\right) ... \mathbb{P}\left(H_n\right)$$

for all $H_1 \in \mathcal{F}_1, ..., H_n \in \mathcal{F}_n$. Let T be an arbitrary index set. Let $\mathcal{F}_t$ be a sub-σ-algebra of $\mathcal{H}$ for each $t \in T$. The collection $\{\mathcal{F}_t : t \in T\}$ is called an *independency* if its every finite subset is an independency [11, p.82].

Moreover, $\mathcal{F}_1, ..., \mathcal{F}_n$ are said to be *conditional independent* given $\mathcal{F}$ if

$$\mathbb{P}_\mathcal{F}\left(H_1 \cap ... \cap H_n\right) = \mathbb{P}_\mathcal{F}\left(H_1\right) ... \mathbb{P}_\mathcal{F}\left(H_n\right)$$

for all $H_1 \in \mathcal{F}_1, ..., H_n \in \mathcal{F}_n$ [11, p.158].

B Causal Effect

In this section, we define what it means for a sub-σ-algebra of the form $\mathcal{H}_S$ to have a *causal effect* on an event $A \in \mathcal{H}$.

Definition B.1. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, $U \in \mathcal{P}(T)$, $A \in \mathcal{H}$ an event and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{H}$ (not necessarily of the form $\mathcal{H}_S$ for some $S \in \mathcal{P}(T)$).

(i) If $K_S(\omega, A) = K_{S\setminus U}(\omega, A)$ for all $S \in \mathcal{P}(T)$ and all $\omega \in \Omega$, then we say that $\mathcal{H}_U$ has *no causal effect on* A, or that $\mathcal{H}_U$ is *non-causal to* A.

We say that $\mathcal{H}_U$ has *no causal effect on* $\mathcal{F}$, or that $\mathcal{H}_U$ is *non-causal to* $\mathcal{F}$, if, for all $A \in \mathcal{F}$, $\mathcal{H}_U$ has no causal effect on A.

(ii) If there exists $\omega \in \Omega$ such that $K_U(\omega, A) \neq \mathbb{P}(A)$, then we say that $\mathcal{H}_U$ has an *active causal effect on* A, or that $\mathcal{H}_U$ is *actively causal to* A.

We say that $\mathcal{H}_U$ has an *active causal effect on* $\mathcal{F}$, or that $\mathcal{H}_U$ is *actively causal to* $\mathcal{F}$, if $\mathcal{H}_U$ has an active causal effect on some $A \in \mathcal{F}$.

(iii) Otherwise, we say that $\mathcal{H}_U$ has a *dormant causal effect on* A, or that $\mathcal{H}_U$ is *dormantly causal to* A.

We say that $\mathcal{H}_U$ has a *dormant causal effect on* $\mathcal{F}$, or that $\mathcal{H}_U$ is *dormantly causal to* $\mathcal{F}$, if $\mathcal{H}_U$ does not have an active causal effect on any event in $\mathcal{F}$ and there exists $A \in \mathcal{F}$ on which $\mathcal{H}_U$ has a dormant causal effect.

Sometimes, we will say that $\mathcal{H}_U$ has a *causal effect* on A to mean that $\mathcal{H}_U$ has either an active or a dormant causal effect on A.

The intuition is as follows. For any $S \in \mathcal{P}(T)$ and any fixed event $A \in \mathcal{H}$, consider the function $\omega_S \mapsto K_S((\omega_{S\cap U}, \omega_{S\setminus U}), A)$. If $\mathcal{H}_U$ has no causal effect on A, then it means that the causal kernel does not depend on the $\omega_{S\cap U}$ component of ω_S. Since this has to hold for all $S \in \mathcal{P}(T)$, it means that it is possible to have, for example, $K_U(\omega, A) = \mathbb{P}(A)$ for all $\omega \in \Omega$ and yet for $\mathcal{H}_U$ to have a causal effect on A. This would be precisely the case where $\mathcal{H}_U$ has a dormant causal effect on A, and it means that, for some $S \in \mathcal{P}(T)$, $\omega_S \mapsto K_S((\omega_{S\cap U}, \omega_{S\setminus U}, A)$ does depend on the $\omega_{S\cap U}$ component.

We collect some straightforward but important special cases in the following remark.

Remark B.2. (a) If $\mathcal{H}_U$ has no causal effect on A, then letting $S = U$ in Definition B.1(i) and applying Definition 2.2(i), we can see that, for all $\omega \in \Omega$,

$$K_U(\omega, A) = K_{U\setminus U}(\omega, A) = K_\emptyset(\omega, A) = \mathbb{P}(A).$$

In particular, this means that $\mathcal{H}_U$ cannot have both no causal effect and active causal effect on A.

(b) It is immediate that the trivial σ-algebra $\mathcal{H}_\emptyset = \{\emptyset, \Omega\}$ has no causal effect on any event $A \in \mathcal{H}$. Conversely, it is also clear that $\mathcal{H}_U$ for any $U \in \mathcal{P}(T)$ has no causal effect on the trivial σ-algebra.

(c) Let $U \in \mathcal{P}(T)$ and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{H}$. If $\mathcal{H}_U \cap \mathcal{F} \neq \{\emptyset, \Omega\}$, then $\mathcal{H}_U$ has an active causal effect on $\mathcal{F}$, since, for $A \in \mathcal{H}_U \cap \mathcal{F}$ with $A \neq \emptyset$ and $A \neq \Omega$, Definition 2.2(ii) tells us that $K_U(\cdot, A) = 1_A(\cdot) \neq \mathbb{P}(A)$. In particular, $\mathcal{H}_U$ has an active causal effect on itself. Further, the full σ-algebra $\mathcal{H} = \mathcal{H}_T$ has an active causal effect on all of its sub-σ-algebras except the trivial σ-algebra, and every $\mathcal{H}_U, U \in \mathcal{P}(T)$ except the trivial σ-algebra has an active causal effect on the full σ-algebra $\mathcal{H}$.

(d) Let $U \in \mathcal{P}(T)$ and $\mathcal{F}_1, \mathcal{F}_2$ be sub-σ-algebras of $\mathcal{H}$. If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{H}_U$ has no causal effect on $\mathcal{F}_2$, then it is clear that $\mathcal{H}_U$ has no causal effect on $\mathcal{F}_1$.

(e) If $\mathcal{H}_U$ has no causal effect on an event A, then for any $V \in \mathcal{P}(T)$ with $V \subseteq U$, $\mathcal{H}_V$ has no causal effect on A. Indeed, take any $S \in \mathcal{P}(T)$. Then using the fact that $\mathcal{H}_U$ has no causal effect on A, see that, for any $\omega \in \Omega$,

$$K_{S\setminus V}(\omega, A) = K_{(S\setminus V)\setminus U}(\omega, A) \qquad \text{applying Definition B.1(i) with } S \setminus V$$

$$
\begin{aligned}
&= K_{S\setminus U}(\omega, A) && \text{since } V \subseteq U \\
&= K_S(\omega, A) && \text{applying Definition B.1(i) with } S.
\end{aligned}
$$

Since $S \in \mathcal{P}(T)$ was arbitrary, we have that $\mathcal{H}_V$ has no causal effect on A.

(f) Contrapositively, if $U, V \in \mathcal{P}(T)$ with $V \subseteq U$ and $\mathcal{H}_V$ has a causal effect on A, then $\mathcal{H}_U$ has a causal effect on A.

(g) If $U \in \mathcal{P}(T)$ has no causal effect on A, then for any $V \in \mathcal{P}(T)$, we have

$$K_V(\omega, A) = K_{U\cup V}(\omega, A).$$

Indeed,

$$
\begin{aligned}
K_{U\cup V}(\omega, A) &= K_{(U\cup V)\setminus(U\setminus V)}(\omega, A) && \text{since } U \setminus V \text{ has no causal effect on } A \text{ by (e)} \\
&= K_V(\omega, A) && \text{since } (U \cup V) \setminus (U \setminus V) = V.
\end{aligned}
$$

(h) If $U, V \in \mathcal{P}(T)$ and neither $\mathcal{H}_U$ nor $\mathcal{H}_V$ has a causal effect on A, then $\mathcal{H}_{U\cup V}$ has no causal effect on A. Indeed, for any $S \in \mathcal{P}(T)$ and any $\omega \in \Omega$,

$$
\begin{aligned}
K_{S\setminus(U\cup V)}(\omega, A) &= K_{(S\setminus U)\setminus V}(\omega, A) \\
&= K_{S\setminus U}(\omega, A) && \text{as } V \text{ has no causal effect on } A \\
&= K_S(\omega, A) && \text{as } U \text{ has no causal effect on } A.
\end{aligned}
$$

Since $S \in \mathcal{P}(T)$ was arbitrary, $\mathcal{H}_{U\cup V}$ has no causal effect on A.

(i) Contrapositively, if $U, V \in \mathcal{P}(T)$ and $\mathcal{H}_{U\cup V}$ has a causal effect on A, then either $\mathcal{H}_U$ or $\mathcal{H}_V$ has a causal effect on A.

Following the definition of no causal effect, we define the notion of a *trivial causal kernel*.

Definition B.3. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T} \mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U \in \mathcal{P}(T)$. We say that the causal kernel K_U is *trivial* if $\mathcal{H}_U$ has no causal effect on $\mathcal{H}_{T\setminus U}$.

Note that we can decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_U \otimes \mathcal{H}_{T\setminus U}$, and so $\mathcal{H}$ is generated by events of the form $A \times B$ for $A \in \mathcal{H}_U$ and $B \in \mathcal{H}_{T\setminus U}$. But if K_U is trivial, then we have, by Axiom 2.2(ii), $K_U(\omega, A \times B) = 1_A(\omega)\mathbb{P}(B)$ for such a rectangle.

We also define a "conditional" version of causal effects.

Definition B.4. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T} \mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, $U, V \in \mathcal{P}(T)$, $A \in \mathcal{H}$ an event and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{H}$ (not necessarily of the form $\mathcal{H}_S$ for some $S \in \mathcal{P}(T)$).

(i) If $K_{S\cup V}(\omega, A) = K_{(S\cup V)\setminus(U\setminus V)}(\omega, A)$ for all $S \in \mathcal{P}(T)$ and all $\omega \in \Omega$, then we say that $\mathcal{H}_U$ has *no causal effect on* A *given* $\mathcal{H}_V$, or that $\mathcal{H}_U$ is *non-causal to* A *given* $\mathcal{H}_V$.

We say that $\mathcal{H}_U$ has *no causal effect on* $\mathcal{F}$ *given* $\mathcal{H}_V$, or that $\mathcal{H}_U$ is *non-causal to* $\mathcal{F}$ *given* $\mathcal{H}_V$, if, for all $A \in \mathcal{F}$, $\mathcal{H}_U$ has no causal effect on A given $\mathcal{H}_V$.

(ii) If there exists $\omega \in \Omega$ such that $K_{U\cup V}(\omega, A) \neq K_V(\omega, A)$, then we say that $\mathcal{H}_U$ has an *active causal effect on* A *given* $\mathcal{H}_V$, or that $\mathcal{H}_U$ is *actively causal to* A *given* $\mathcal{H}_V$.

We say that $\mathcal{H}_U$ has an *active causal effect on* $\mathcal{F}$ *given* $\mathcal{H}_V$, or that $\mathcal{H}_U$ is *actively causal to* $\mathcal{F}$ *given* $\mathcal{H}_V$, if $\mathcal{H}_U$ has an active causal effect on some $A \in \mathcal{F}$.

(iii) Otherwise, we say that $\mathcal{H}_U$ has a *dormant causal effect on* A *given* $\mathcal{H}_V$, or that $\mathcal{H}_U$ is *dormantly causal to* A *given* $\mathcal{H}_V$.

We say that $\mathcal{H}_U$ has a *dormant causal effect on* $\mathcal{F}$ *given* $\mathcal{H}_V$, or that $\mathcal{H}_U$ is *dormantly causal to* $\mathcal{F}$ *given* $\mathcal{H}_V$, if $\mathcal{H}_U$ does not have an active causal effect on any event in $\mathcal{F}$ given $\mathcal{H}_V$ and there exists $A \in \mathcal{F}$ on which $\mathcal{H}_U$ has a dormant causal effect given $\mathcal{H}_V$.

Sometimes, we will say that $\mathcal{H}_U$ has a *causal effect on* A *given* $\mathcal{H}_V$ to mean that $\mathcal{H}_U$ has either an active or a dormant causal effect on A given $\mathcal{H}_V$.

The intuition is as follows. For any fixed $S \in \mathcal{P}(T)$ and any fixed event $A \in \mathcal{H}$. consider the function $\omega_{S\cup V} \mapsto K_{S\cup V}((\omega_{(S\cup V)\setminus(U\setminus V)}, \omega_{S\cap(U\setminus V)}), A)$. If $\mathcal{H}_U$ has no causal effect on A given $\mathcal{H}_V$, then it means that the causal kernel does not depend on the $\omega_{S\cap(U\setminus V)}$ component of $\omega_{S\cup V}$; in other words, $\mathcal{H}_U$ only has an influence on A through its V component.

We collect some important special cases in the following remark.

Remark B.5. (a) Letting $V = U$, we always have $K_{S\cup U}(\omega, A) = K_{(S\cup U)\setminus(U\setminus U)}(\omega, A) = K_{S\cup U}(\omega, A)$ for all $\omega \in \Omega$ and $A \in \mathcal{H}$, which means that $\mathcal{H}_U$ has no causal effect on any event $A \in \mathcal{H}$ given itself.

(b) If $\mathcal{H}_U$ has no causal effect on A given $\mathcal{H}_V$, then letting $U = S$ in Definition B.4(i), we see that, for all $\omega \in \Omega$,

$$K_{U\cup V}(\omega, A) = K_V(\omega, A).$$

In particular, this means that $\mathcal{H}_U$ cannot have both no causal effect and active causal effect on A given $\mathcal{H}_V$.

(c) The case $V = \emptyset$ reduces Definition B.4 to Definition B.1, i.e. $\mathcal{H}_U$ having no causal effect in the sense of Definition B.1 is the same as $\mathcal{H}_U$ having no causal effect given $\{\emptyset, \Omega\}$ in the sense of Definition B.4, etc.

(d) It is possible for $\mathcal{H}_U$ to be causal to an event A, and for there to exist $V \in \mathcal{P}(T)$ such that $\mathcal{H}_U$ has no causal effect on A given $\mathcal{H}_V$. However, if $\mathcal{H}_U$ has no causal effect on A, then for any $V \in \mathcal{P}(T)$, $\mathcal{H}_U$ has no causal effect on A given $\mathcal{H}_V$. To see this, note that Remark B.2(e) tells us that $U \setminus V$ also does not have any causal effect on A. Then given any $S \in \mathcal{P}(T)$,

$$K_{S\cup V}(\omega, A) = K_{(S\cup V)\setminus(U\setminus V)}(\omega, A),$$

applying Definition B.1(i) to $S \cup V$. Since $S \in \mathcal{P}(T)$ was arbitrary, $\mathcal{H}_U$ has no causal effect on A given $\mathcal{H}_V$.

C Interventions

In this section, we provide a few more definitions and results related to the notion of interventions, introduced in Definition 2.3.

First, we make a few remarks on how the intervention causal kernels $K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}$ behave in some special cases, depending on the relationship between U and S.

Remark C.1. (a) For $S \in \mathcal{P}(T)$ with $U \subseteq S$, we have, for all $\omega \in \Omega$ and all $A \in \mathcal{H}$,

$$\begin{aligned} K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A) &= \int L_U(\omega_U, d\omega'_U) K_S((\omega_{S\setminus U}, \omega'_U), A) \\ &= \int \delta_{\omega_U}(d\omega'_U) K_S((\omega_{S\setminus U}, \omega'_U), A) \qquad \text{by Definition 2.2(ii)} \\ &= K_S((\omega_{S\setminus U}, \omega_U), A) \\ &= K_S(\omega, A). \end{aligned}$$

This means that, after an intervention on $\mathcal{H}_U$, subsequent interventions on $\mathcal{H}_S$ with $\mathcal{H}_U \subseteq \mathcal{H}_S$ simply overwrite the original intervention. Note that this is reminiscent of the "partial ordering on the set of interventions" in [44], but in our setting, this is given by the partial ordering induced by the inclusion structure of sub-σ-algebras of $\mathcal{H}$.

(b) For $S \in \mathcal{P}(T)$ with $S \subseteq U$,

$$K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A) = \int L_S(\omega_S, d\omega'_U) K_U(\omega'_U, A)$$

for all $\omega \in \Omega$ and $A \in \mathcal{H}$, i.e. $K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}$ is a product of the two kernels K_U and L_S [11, p.39]; in particular, $K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A) = L_S(\omega, A)$ for all $A \in \mathcal{H}_U$.

(c) For $S \in \mathcal{P}(T)$ with $S \cap U = \emptyset$,

$$\begin{aligned} K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A) &= \int L_\emptyset(\omega_\emptyset, d\omega'_U) K_{S\cup U}((\omega_S, \omega'_U), A) \\ &= \int \mathbb{Q}(d\omega'_U) K_{S\cup U}((\omega_S, \omega'_U), A) \qquad \text{by Definition 2.2(i)} \end{aligned}$$

for all $\omega \in \Omega$ and $A \in \mathcal{H}$, i.e. the effect of intervening on $\mathcal{H}_U$ with $\mathbb{Q}$ then $\mathcal{H}_S$ is the same as intervening on $\mathcal{H}_{U\cup S}$ with a product measure of $\mathbb{Q}$ on $\mathcal{H}_U$ and whatever measure we place on $\mathcal{H}_S$.

We give it a name for the special case in which the internal causal kernels are all trivial (see Definition B.3).

Definition C.2. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T} \mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U \in \mathcal{P}(T)$ and $\mathbb{Q}$ a probability measure on $(\Omega, \mathcal{H}_U)$. A *hard intervention on* $\mathcal{H}_U$ *via* $\mathbb{Q}$ is a new causal space $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(U,\mathbb{Q})}, \mathbb{K}^{\text{do}(U,\mathbb{Q},\text{hard})})$, where the intervention measure $\mathbb{P}^{\text{do}(U,\mathbb{Q})}$ is a probability measure $(\Omega, \mathcal{H})$ defined in the same way as in Definition 2.3, and the intervention causal mechanism $\mathbb{K}^{\text{do}(U,\mathbb{Q},\text{hard})} = \{K_S^{\text{do}(U,\mathbb{Q},\text{hard})} : S \in \mathcal{P}(T)\}$ consists of causal kernels that are obtained from the intervention causal kernels in Definition 2.3 in which $L_{S\cap U}$ is a trivial causal kernel, i.e. one that has no causal effect on $\mathcal{H}_{U\setminus S}$.

From the discussion following Definition B.3, we have that, for $A \in \mathcal{H}_{S\cap U}$ and $B \in \mathcal{H}_{U\setminus S}$, $L_{S\cap U}(\omega, A \times B) = 1_A(\omega_{S\cap U})\mathbb{Q}(B)$.

The next result gives an explicit expression for the causal kernels obtained after a hard intervention.

Theorem C.3. *Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T} \mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U \in \mathcal{P}(T)$ and $\mathbb{Q}$ a probability measure on $(\Omega, \mathcal{H}_U)$. Then after a hard intervention on $\mathcal{H}_U$ via $\mathbb{Q}$, the intervention causal kernels $K_S^{\text{do}(U,\mathbb{Q},\text{hard})}$ are given by*

$$K_S^{\text{do}(U,\mathbb{Q},\text{hard})}(\omega, A) = K_S^{\text{do}(U,\mathbb{Q},\text{hard})}(\omega_S, A) = \int \mathbb{Q}(d\omega'_{U\setminus S}) K_{S\cup U}((\omega_S, \omega'_{U\setminus S}), A).$$

Intuitively, hard interventions do not encode any internal causal relationships within $\mathcal{H}_U$, so after we subsequently intervene on $\mathcal{H}_S$, the measure $\mathbb{Q}$ that we originally imposed on $\mathcal{H}_U$ remains on $\mathcal{H}_{U\setminus S}$.

The following lemma contains a couple of results about particular sub-σ-algebras having no causal effects on particular events in the intervention causal space, regardless of the measure and causal mechanism that was used for the intervention.

Lemma C.4. *Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T} \mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U \in \mathcal{P}(T)$, $\mathbb{Q}$ a probability measure on $(\Omega, \mathcal{H}_U)$ and $\mathbb{L} = \{L_V : V \in \mathcal{P}(U)\}$ a causal mechanism on $(\Omega, \mathcal{H}_U, \mathbb{Q})$. Suppose we intervene on $\mathcal{H}_U$ via $(\mathbb{Q}, \mathbb{L})$.*

(i) For $A \in \mathcal{H}_U$ and $V \in \mathcal{P}(T)$ with $V \cap U = \emptyset$, $\mathcal{H}_V$ has no causal effect on A (c.f. Definition B.1(i)) in the intervention causal space $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(U,\mathbb{Q})}, \mathbb{K}^{\text{do}(U,\mathbb{Q},\mathbb{L})})$, i.e. events in the σ-algebra $\mathcal{H}_U$ on which intervention took place are not causally affected by σ-algebras outside $\mathcal{H}_U$.

(ii) Again, let $V \in \mathcal{P}(T)$ with $V \cap U = \emptyset$, and also let $A \in \mathcal{H}$ be any event. If, in the original causal space, $\mathcal{H}_V$ had no causal effect on A, then in the intervention causal space, $\mathcal{H}_V$ has no causal effect on A either.

(iii) Now let $V \in \mathcal{P}(T)$, $A \in \mathcal{H}$ any event and suppose that the intervention on $\mathcal{H}_U$ via $\mathbb{Q}$ is hard. Then if $\mathcal{H}_V$ had no causal effect on A in the original causal space, then $\mathcal{H}_V$ has no causal effect on A in the intervention causal space either.

Lemma C.4(ii) and (iii) tell us that, if $\mathcal{H}_V$ had no causal effect on A in the original causal space, then by intervening on $\mathcal{H}_U$ with $V \cap U = \emptyset$ or by any hard intervention, we cannot create a causal effect from $\mathcal{H}_v$ on A. However, by intervening on a sub-σ-algebra that contains both $\mathcal{H}_V$ and (a part of) A, and manipulating the internal causal mechanism $\mathbb{L}$ appropriately, it is clear that we can create a causal effect from $\mathcal{H}_V$.

The next result tells us that if a sub-σ-algebra $\mathcal{H}_U$ has a dormant causal effect on an event A, then there is a sub-σ-algebra of $\mathcal{H}_U$ and a hard intervention after which that sub-σ-algebra has an active causal effect on A.

Lemma C.5. *Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U \in \mathcal{P}(T)$. For an event $A \in \mathcal{H}$, if $\mathcal{H}_U$ has a dormant causal effect on A in the original causal space, then there exists a hard intervention and a subset $V \subseteq U$ such that in the intervention causal space, $\mathcal{H}_V$ has an active causal effect on A.*

The next result is about what happens to a causal effect of a sub-σ-algebra that has no causal effect on an event conditioned on another sub-σ-algebra, after intervening on that sub-σ-algebra.

Lemma C.6. *Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U, V \in \mathcal{P}(T)$. For an event $A \in \mathcal{H}$, suppose that $\mathcal{H}_U$ has no causal effect on A given $\mathcal{H}_V$ (see Definition B.4). Then after an intervention on $\mathcal{H}_V$ via any $(\mathbb{Q}, \mathbb{L})$, $\mathcal{H}_{U\setminus V}$ has no causal effect on A.*

The next result shows that, under a hard intervention, a time-respecting causal mechanism stays time-respecting.

Theorem C.7. *Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, where the index set T can be written as $T = W \times \tilde{T}$, with W representing time and $\mathbb{K}$ respecting time. Take any $U \in \mathcal{P}(T)$ and any probability measure $\mathbb{Q}$ on $\mathcal{H}_U$. Then the intervention causal mechanism $\mathbb{K}^{\text{do}(U,\mathbb{Q},\text{hard})}$ also respects time.*

D Sources

In causal spaces, the observational distribution $\mathbb{P}$ and the causal mechanism $\mathbb{K}$ are completely decoupled. In Section 3.1, we give a detailed argument as to why this is desirable, but of course, there is no doubt that the special case in which the causal kernels coincide with conditional measures with respect to $\mathbb{P}$ is worth studying. To that end, we introduce the notion of *sources*.

Definition D.1. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, $U \in \mathcal{P}(T)$, $A \in \mathcal{H}$ an event and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{H}$. We say that $\mathcal{H}_U$ is a *(local) source* of A if $K_U(\cdot, A)$ is a version of the conditional probability $\mathbb{P}_{\mathcal{H}_U}(A)$. We say that $\mathcal{H}_U$ is a *(local) source* of $\mathcal{F}$ if $\mathcal{H}_U$ is a source of all $A \in \mathcal{F}$. We say that $\mathcal{H}_U$ is a *global source* of the causal space if $\mathcal{H}_U$ is a source of all $A \in \mathcal{H}$.

Clearly, source σ-algebras are not unique (whether local or global). It is easy to see that $\mathcal{H}_\emptyset = \{\emptyset, \Omega\}$ and $\mathcal{H} = \mathcal{H}_T = \otimes_{t\in T}\mathcal{E}_t$ are global sources, and axiom (ii) of Definition 2.2 implies that any $\mathcal{H}_S$ is a local source of any of its sub-σ-algebras, including itself, since, for any $A \in \mathcal{H}_U$, $\mathbb{P}_{\mathcal{H}_U}(A) = 1_A$. Also, a sub-$\sigma$-algebra of a source is not necessarily a source, nor is a σ-algebra that contains a source necessarily a source (whether local or global). In Example 2.5 above, altitude is a source of temperature (and hence a global source), since the causal kernel corresponding to temperature coincides with the conditional measure given altitude, but temperature is not a source of altitude.

When we intervene on $\mathcal{H}_U$ (via any $(\mathbb{Q}, \mathbb{L})$), $\mathcal{H}_U$ becomes a global source. This precisely coincides with the "gold standard" that is randomised control trials in causal inference, i.e. the idea that, if we are able to intervene on $\mathcal{H}_U$, then the causal effect of $\mathcal{H}_U$ on any event can be obtained by first intervening on $\mathcal{H}_U$, then considering the conditional distribution on $\mathcal{H}_U$. Next is a theorem showing that when one intervenes on $\mathcal{H}_U$, then $\mathcal{H}_U$ becomes a source.

Theorem D.2. *Suppose we have a causal space $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$, and let $U \in \mathcal{P}(T)$.*

(i) For any measure $\mathbb{Q}$ on $\mathcal{H}_U$ and any causal mechanism $\mathbb{L}$ on $(\Omega, \mathcal{H}_U, \mathbb{Q})$, the causal kernel $K_U^{\text{do}(U,\mathbb{Q},\mathbb{L})} = K_U$ is a version of $\mathbb{P}_{\mathcal{H}_U}^{\text{do}(U,\mathbb{Q})}$, which means that $\mathcal{H}_U$ is a global source σ-algebra of the intervened causal space $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(U,\mathbb{Q})}, \mathbb{K}^{\text{do}(U,\mathbb{Q},\mathbb{L})})$.

(ii) Suppose $V \in \mathcal{P}(T)$ with $V \subseteq U$. Suppose that the measure $\mathbb{Q}$ on $(\Omega, \mathcal{H}_U)$ factorises over $\mathcal{H}_V$ and $\mathcal{H}_{U\setminus V}$, i.e. for any $A \in \mathcal{H}_V$ and $B \in \mathcal{H}_{U\setminus V}$, $\mathbb{Q}(A \cap B) = \mathbb{Q}(A)\mathbb{Q}(B)$. Then after a hard intervention on $\mathcal{H}_U$ via $\mathbb{Q}$, the causal kernel $K_V^{\text{do}(U,\mathbb{Q},\text{hard})}$ is a version of $\mathbb{P}_V^{\text{do}(U,\mathbb{Q})}$, which means that $\mathcal{H}_V$ is a global source σ-algebra of the intervened causal space $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(U,\mathbb{Q})}, \mathbb{K}^{\text{do}(U,\mathbb{Q},\text{hard})})$.

Let $A \in \mathcal{H}$ be an event, and $U \in \mathcal{P}(T)$. By the definition of the intervention measure (Definition 2.3), we always have

$$\mathbb{P}^{\text{do}(U,\mathbb{Q})}(A) = \int \mathbb{Q}(d\omega) K_U(\omega, A),$$

hence $\mathbb{P}^{\text{do}(U,\mathbb{Q})}(A)$ can be written in terms of $\mathbb{P}$ and $\mathbb{Q}$ if $K_U(\omega, A)$ can be written in terms of $\mathbb{P}$. This can be seen to occur in three trivial cases: first, if $\mathcal{H}_U$ is a local source of A (see Definition D.1), in which case $K_U(\omega, A) = \mathbb{P}_{\mathcal{H}_U}(\omega, A)$; secondly, if $\mathcal{H}_U$ has no causal effect on A (see Definition B.1), in which case $K_U(\omega, A) = \mathbb{P}(A)$; and finally, if $A \in \mathcal{H}_U$, in which case, by intervention determinism (Definition 2.2(ii), we have $K_U(\omega, A) = 1_A(\omega)$. In the latter case, we do not even have dependence on $\mathbb{P}$. Can we generalise these results?

Lemma D.3. *Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t \in T} E_t, \otimes_{t \in T} \mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space. Let $A \in \mathcal{H}$ be an event, and $U \in \mathcal{P}(T)$. If there exists a sub-σ-algebra $\mathcal{G}$ of $\mathcal{H}$ (not necessarily of the form $\mathcal{H}_V$ for some $V \in \mathcal{P}(T)$) such that*

(i) the conditional probability $\mathbb{P}^{\text{do}(U,\mathbb{Q})}_{\mathcal{H}_U \vee \mathcal{G}}(\cdot, A)$ can be written in terms of $\mathbb{P}$ and $\mathbb{Q}$;

(ii) the causal kernel $K_U(\cdot, B)$ can be written in terms of $\mathbb{P}$ for all $B \in \mathcal{G}$;

then $\mathbb{P}^{\text{do}(U,\mathbb{Q})}(A)$ can be written in terms of $\mathbb{P}$ and $\mathbb{Q}$.

Remark D.4. The three cases discussed in the paragraph above Lemma D.3 are special cases of the Lemma with $\mathcal{G}$ being any sub-σ-algebra of $\mathcal{H}$ with $\{\emptyset, \Omega\} \subseteq \mathcal{G} \subseteq \mathcal{H}_U$. In this case, condition (ii) is trivially satisfied since we have $K_U(\cdot, B) = 1_B(\cdot)$ by intervention determinism (Definition 2.2(ii)), and for condition (i), by Theorem D.2(i), we have $\mathbb{P}^{\text{do}(U,\mathbb{Q})}_{\mathcal{H}_U}(\cdot, A) = K_U(\cdot, A)$, which means that the problem reduces to checking if $K_U(\cdot, A)$ can be written in terms of $\mathbb{P}$.

Proof. By law of total expectations, for any $V \in \mathcal{P}(T)$, we have

$$\begin{aligned}\mathbb{P}^{\text{do}(U,\mathbb{Q})}(A) &= \int \mathbb{P}^{\text{do}(U,\mathbb{Q})}_{\mathcal{H}_U \vee \mathcal{G}}(\omega, A) \mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega) \\ &= \int \mathbb{P}^{\text{do}(U,\mathbb{Q})}_{\mathcal{H}_U \vee \mathcal{G}}(\omega, A) \int \mathbb{Q}(d\omega') K_U(\omega', d\omega).\end{aligned}$$

Here, $\mathbb{P}^{\text{do}(U,\mathbb{Q})}_{\mathcal{H}_U \vee \mathcal{G}}(\omega, A)$ can be written in terms of $\mathbb{P}$ and $\mathbb{Q}$ by condition (i). Moreover, note that it suffices to be able to write the restriction of $K_U(\omega', \cdot)$ to $\mathcal{H}_U \vee \mathcal{G}$ in terms of $\mathbb{P}$, since the integration is of a $\mathcal{H}_U \vee \mathcal{G}$-measurable function. Since the collection of intersections $\{D \cap B, D \in \mathcal{H}_U, B \in \mathcal{G}\}$ is a π-system that generates $\mathcal{H}_U \vee \mathcal{G}$ [11, p.5, 1.18], it suffices to check that $K_U(\omega', D \cap B)$ can be written in terms of $\mathbb{P}$ for all $D \in \mathcal{H}_U$ and $B \in \mathcal{G}$. But by interventional determinism (Definition 2.2(ii)), we have $K_U(\omega', D \cap B) = 1_D(\omega') K_U(\omega', B)$. Since $K_U(\omega', B)$ can be written in terms of $\mathbb{P}$ by condition (ii), the restriction of $K_U(\omega', \cdot)$ to $\mathcal{H}_U \vee \mathcal{G}$ can be written in terms of $\mathbb{P}$, and hence $\mathbb{P}^{\text{do}(U,\mathbb{Q})}(A)$ can be written in terms of $\mathbb{P}$ and $\mathbb{Q}$. □

Corollary D.5. *Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t \in T} E_t, \otimes_{t \in T} \mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space. Let $A \in \mathcal{H}$ be an event, and $U \in \mathcal{P}(T)$. If there exists a $V \in \mathcal{P}(T)$ such that condition (i) of Lemma D.3 is satisfied with $\mathcal{G} = \mathcal{H}_V$ and one of the following conditions is satisfied:*

(a) $\mathcal{H}_U$ is a local source of $\mathcal{H}_V$; or

(b) $\mathcal{H}_U$ has no causal effect on $\mathcal{H}_V$; or

(c) $V \subseteq U$,

then $\mathbb{P}^{\text{do}(U,\mathbb{Q})}(A)$ can be written in terms of $\mathbb{P}$ and $\mathbb{Q}$.

Proof. Condition (i) of Lemma D.3 is satisfied by hypothesis. If one of (a), (b) or (c) is satisfied, then trivially, condition (ii) of Lemma D.3 is also satisfied. The result now follows from Lemma D.3. □

The above is reminiscent of "valid adjustments" in the context of structural causal models [42, p.115, Proposition 6.41], and in fact contains the valid adjustments.

E Counterfactuals

There are various notions of counterfactuals in the literature. The one considered in the SCM literature is the *interventional counterfactual*, which captures the notion of "what would have happened if we intervened on the space, given some observations (that are possibly contradictory to the intervention we imagine we would have done)". Recently, *backtracking counterfactuals* have also been integrated into the SCM framework [53]. This captures the notion of "what would have happened if background conditions of the world had been different, given that the causal laws of the system stay the same?" Finally, we note that in the potential outcomes framework, the random variables representing "potential outcomes" that form the primitives of the framework can be directly counterfactual.

Vanilla probability measures have just one argument, i.e. the event. Conditional measures and causal kernels (in the sense of our Definition 2.2) have two arguments, the first being the outcome which we either observe or force the occurrence of, and the second being the event in whose measure we are interested. For both of the above concepts of counterfactuals, we need to go one step further and consider three arguments. The first is the outcome which we observe, just like in conditioning, and the last should be the event in whose measure we are interested. For interventional counterfactuals, the second argument should be an outcome which we imagine to have forced the occurrence of given that we observed the outcome of the first argument, and for backtracking counterfactuals, the second argument should be an outcome which we imagine to have observed instead of the outcome in the first argument which we actually observed.

From these principles, we tentatively propose to extend Definition 2.2 to account for *interventional counterfactuals* as follows.

Definition E.1. A *causal space* is defined as the quadruple $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K})$, where $(\Omega, \mathcal{H}, \mathbb{P}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P})$ is a probability space and $\mathbb{K} = \{K_{S,\mathcal{F}} : S \in \mathcal{P}(T), \mathcal{F} \text{ sub-}\sigma\text{-algebra of } \mathcal{H}\}$, called the *causal mechanism*, is a collection of functions $K_{S,\mathcal{F}} : \Omega \times \Omega \times \mathcal{H} \to [0, 1]$, called the *causal kernel on* $\mathcal{H}_S$ *after observing* $\mathcal{F}$, such that

(i) for each fixed $\eta \in \Omega$ and $A \in \mathcal{H}$, $K_{S,\mathcal{F}}(\cdot, \eta, A)$ is measurable with respect to $\mathcal{H}_S$;

(ii) for each fixed $\omega \in \Omega$ and $A \in \mathcal{H}$, $K_{S,\mathcal{F}}(\omega, \cdot, A)$ is measurable with respect to $\mathcal{F}$;

(iii) for each fixed pair $(\omega, \eta) \in \Omega \times \Omega$, $K_{S,\mathcal{F}}(\omega, \eta, \cdot)$ is a measure on $\mathcal{H}$;

(iv) for all $A \in \mathcal{H}$ and $\omega, \eta \in \Omega$,
$$K_{\emptyset,\mathcal{F}}(\omega, \eta, A) = \mathbb{P}_{\mathcal{F}}(\eta, A);$$

(v) for all $A \in \mathcal{H}_S$, all $B \in \mathcal{H}$ and all $\omega, \eta \in \Omega$,
$$K_{S,\mathcal{F}}(\omega, \eta, A \cap B) = 1_A(\omega) K_S(\omega, \eta, B);$$
in particular, for $A \in \mathcal{H}_S$, $K_{S,\mathcal{F}}(\omega, \eta, A) = 1_A(\omega) K_{S,\mathcal{F}}(\omega, \eta, \Omega) = 1_A(\omega)$;

(vi) for all $A \in \mathcal{H}$, $\omega \in \Omega$ and sub-σ-algebras $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$,
$$\mathbb{E}_{\mathcal{F}}\left[K_{S,\mathcal{G}}(\omega, \cdot, A)\right] = K_{S,\mathcal{F}}(\omega, \cdot, A).$$

Note that letting $\mathcal{F} = \{\emptyset, \Omega\}$ trivially recovers the causal space as defined in Definition 2.2. Moreover, letting $S = \emptyset$, we recover the conditional distribution given $\mathcal{F}$.

Recall that the one of the biggest philosophical differences between the SCM framework and our proposed causal spaces (Definition 2.2) was the fact that SCMs start with the variables, the structural equations and the noise distributions as the *primitive objects*, and the observational and interventional distributions over the endogenous variables are *derived* from these, whereas causal spaces take the observational and interventional distributions as the *primitive objects* (the latter via causal kernels). Note that, in the above extended definition of causal spaces incorporating interventional counterfactuals (Definition E.1), we applied the same principles, in that we treated the observational distribution ($\mathbb{P}$), interventional distributions ($K_{S,\{\emptyset,\Omega\}}$) and the (interventional) counterfactual distributions ($K_{S,\mathcal{F}}$) as the primitive objects.

This differs significantly from the SCM framework, where again, the (interventional) counterfactual distributions are *derived* from the structural equations, by first conditioning on the observed values of the endogenous variables to get a modified (often Dirac) measure on the exogenous variables, then intervening on some of the endogenous variables, deriving the measure on the rest of the endogenous variables by propagating these through the same structural equations. We see the value in this approach in that the (interventional) counterfactual distributions can be neatly derived from the same primitive objects that are used to calculate the observational and interventional distribution. However, we argue that this cannot be an *axiomatisation* of (interventional) counterfactual distributions in the strictest sense, because it relies on assumptions. In particular, it strongly relies on the assumption that the endogenous variables have no causal effect on the exogenous variables, and when this assumption is violated, i.e. when there is a hidden mediator, calculation of (interventional) counterfactual distributions is not possible. In contrast, Definition E.1 treat the (interventional) counterfactual measures as the primitive objects, and does not impose any a priori assumptions about the system.

As mentioned in Section 5 of the main body of the paper, we leave further developments of this interventional counterfactual causal space, as well as the definition of backtracking counterfactual causal space, as essential future work.

F Proofs

Theorem 2.6. From Definition 2.3, $\mathbb{P}^{\text{do}(U,\mathbb{Q})}$ is indeed a measure on $(\Omega,\mathcal{H})$, and $\mathbb{K}^{\text{do}(U,\mathbb{Q},\mathbb{L})}$ is indeed a valid causal mechanism on $(\Omega,\mathcal{H},\mathbb{P}^{\text{do}(U,\mathbb{Q})})$, i.e. they satisfy the axioms of Definition 2.2.

Proof. That $\mathbb{P}^{\text{do}(U,\mathbb{Q})}$ is a measure on $(\Omega,\mathcal{H})$ follows immediately from the usual construction of measures from measures and transition probability kernels, see e.g. Çınlar [11, p.38, Theorem 6.3]. It remains to check that $\mathbb{K}^{\text{do}(U,\mathbb{Q},\mathbb{L})}$ is a valid causal mechanism in the sense of Definition 2.2.

(i) For all $A\in\mathcal{H}$ and $\omega\in\Omega$,

$$\begin{aligned}K_{\emptyset}^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega,A)&=\int L_{\emptyset}(\omega_{\emptyset},d\omega'_U)K_U((\omega_{\emptyset},\omega'_U),A)\\&=\int \mathbb{Q}(d\omega')K_U(\omega',A)\\&=\mathbb{P}^{\text{do}(U,\mathbb{Q})}(A),\end{aligned}$$

where we applied Axiom 2.2(i) to $L_{\emptyset}$.

(ii) For all $A\in\mathcal{H}_S$ and $B\in\mathcal{H}$, we have, by Axiom 2.2(ii) using the fact that $A\in\mathcal{H}_S\subseteq\mathcal{H}_{S\cup U}$,

$$\begin{aligned}&K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega,A\cap B)\\&=\int L_{S\cap U}(\omega_{S\cap U},d\omega'_U)K_{S\cup U}((\omega_{S\setminus U},\omega'_U),A\cap B)\\&=\int L_{S\cap U}(\omega_{S\cap U},d\omega'_U)1_A((\omega_{S\setminus U},\omega'_U))K_{S\cup U}((\omega_{S\setminus U},\omega'_U),B)\\&=\int L_{S\cap U}(\omega_{S\cap U},d\omega'_U)1_A((\omega_{S\setminus U},\omega'_{S\cap U}))K_{S\cup U}((\omega_{S\setminus U},\omega'_U),B),\end{aligned}$$

where, in going from the third line to the fourth, we split the ω'_U in $1_A((\omega_{S\setminus U},\omega'_U))$ into components $(\omega'_{S\cap U},\omega'_{U\setminus S})$ and notice that since $A\in\mathcal{H}_S$, 1_A does not depend on the component $\omega'_{U\setminus S}$. Here, the map $\omega'_{S\cap U}\mapsto 1_A((\omega_{S\setminus U},\omega'_{S\cap U}))$ is $\mathcal{H}_{S\cap U}$-measurable, so we can write it as the limit of an increasing sequence of positive $\mathcal{H}_{S\cap U}$-simple functions (see Section A.1), say $(f_n)_{n\in\mathbb{N}}$ with $f_n=\sum_{i_n=1}^{m_n}b_{i_n}1_{B_{i_n}}$, where $B_{i_n}\in\mathcal{H}_{S\cap U}$. Likewise, the map $\omega'_U\mapsto K_{S\cup U}((\omega_{S\setminus U},\omega'_U),B)$ is $\mathcal{H}_U$-measurable, so we can write it as the limit of an increasing sequence of positive $\mathcal{H}_U$-simple functions, say $(g_n)_{n\in\mathbb{N}}$ with $g_n=\sum_{j_n=1}^{l_n}c_{j_n}1_{C_{j_n}}$, where $C_{j_n}\in\mathcal{H}_U$. Hence

$$K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega,A\cap B)=\int L_{S\cap U}(\omega_{S\cap U},d\omega'_U)\left(\lim_{n\to\infty}f_n(\omega'_{S\cap U})\right)\left(\lim_{n\to\infty}g_n(\omega'_U)\right).$$

Since, for each ω'_U, both of the limits exist by construction, namely the original measurable functions, we have that the product of the limits is the limit of the products:

$$K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A\cap B) = \int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U) \lim_{n\to\infty} \left(f_n(\omega'_{S\cap U})g_n(\omega'_U)\right).$$

Here, since f_n and g_n were individually sequences of increasing functions, the pointwise products $f_n g_n$ also form an increasing sequence of functions. Hence, we can apply the monotone convergence theorem to see that

$$\begin{aligned}
&K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A\cap B)\\
&= \lim_{n\to\infty}\int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U) f_n(\omega'_{S\cap U}) g_n(\omega'_U)\\
&= \lim_{n\to\infty}\sum_{i_n=1}^{m_n}\sum_{j_n=1}^{l_n} b_{i_n}c_{j_n}\int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U)1_{B_{i_n}}(\omega'_{S\cap U})1_{C_{j_n}}(\omega'_U)\\
&= \lim_{n\to\infty}\sum_{i_n=1}^{m_n}\sum_{j_n=1}^{l_n} b_{i_n}c_{j_n} L_{S\cap U}(\omega_{S\cap U}, B_{i_n}\cap C_{j_n})\\
&= \lim_{n\to\infty}\sum_{i_n=1}^{m_n}\sum_{j_n=1}^{l_n} b_{i_n}c_{j_n} 1_{B_{i_n}}(\omega_{S\cap U}) L_{S\cap U}(\omega_{S\cap U}, C_{j_n})\\
&= \lim_{n\to\infty}\sum_{i_n=1}^{m_n} b_{i_n} 1_{B_{i_n}}(\omega_{S\cap U}) \sum_{j_n=1}^{l_n} c_{j_n} L_{S\cap U}(\omega_{S\cap U}, C_{j_n})\\
&= \left(\lim_{n\to\infty}\sum_{i_n=1}^{m_n} b_{i_n} 1_{B_{i_n}}(\omega_{S\cap U})\right)\left(\lim_{n\to\infty}\sum_{j_n=1}^{l_n} c_{j_n} L_{S\cap U}(\omega_{S\cap U}, C_{j_n})\right)\\
&= \left(\lim_{n\to\infty} f_n(\omega_{S\cap U})\right)\left(\lim_{n\to\infty}\int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U)\sum_{j_n=1}^{l_n} c_j 1_{C_{j_n}}(\omega'_U)\right)\\
&= 1_A((\omega_{S\setminus U}, \omega_{S\cap U}))\int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U)\lim_{n\to\infty} g_n(\omega'_U)\\
&= 1_A(\omega_S)\int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U) K_{S\cup U}((\omega_{S\setminus U}, \omega'_U), B)\\
&= 1_A(\omega_S) K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega_S, B)
\end{aligned}$$

where, from the fourth line to the fifth, we used Axiom 2.2(ii); from the sixth line to the seventh, we used that limit of the products is the product of the limits again, noting that both of the limits exist by construction; from the eighth line to the ninth, we used monotone convergence theorem again. This is the required result.

□

Theorem C.3. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U\in\mathcal{P}(T)$ and $\mathbb{Q}$ a probability measure on $(\Omega, \mathcal{H}_U)$. Then after a hard intervention on $\mathcal{H}_U$ via $\mathbb{Q}$, the intervention causal kernels $K_S^{\text{do}(U,\mathbb{Q},\text{hard})}$ are given by

$$K_S^{\text{do}(U,\mathbb{Q},\text{hard})}(\omega, A) = K_S^{\text{do}(U,\mathbb{Q},\text{hard})}(\omega_S, A) = \int \mathbb{Q}(d\omega'_{U\setminus S}) K_{S\cup U}((\omega_S, \omega'_{U\setminus S}), A).$$

Proof. We decompose $\mathcal{H}_U$ as a product σ-algebra into $\mathcal{H}_{S\cap U}\otimes\mathcal{H}_{U\setminus S}$. Then events of the form $B\cap C$ with $B\in\mathcal{H}_{S\cap U}$ and $C\in\mathcal{H}_{U\setminus S}$ generate $\mathcal{H}_U$, so for fixed $\omega_{S\cap U}$, the measure $L_{S\cap U}(\omega_{S\cap U}, \cdot)$ is completely determined by $L_{S\cap U}(\omega_{S\cap U}, B\cap C)$ for all $B\in\mathcal{H}_{S\cap U}$, $C\in\mathcal{H}_{U\setminus S}$. But we have

$$L_{S\cap U}(\omega_{S\cap U}, B\cap C) = \delta_{\omega_{S\cap U}}(B) L_{S\cap U}(\omega_{S\cap U}, C) \qquad \text{by Axiom 2.2(ii)}$$

$$= \delta_{\omega_{S\cap U}}(B)\mathbb{Q}(C),$$

since $L_{S\cap U}$ is trivial and $C \in \mathcal{H}_{U\setminus S}$. So the measure $L_{S\cap U}(\omega_{S\cap U}, \cdot)$ is a product measure of $\delta_{\omega_{S\cap U}}$ and $\mathbb{Q}$. Hence, applying Fubini's theorem,

$$\begin{aligned}
K_S^{\text{do}(U,\mathbb{Q},\text{hard})}(\omega, A) &= \int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U)K_{S\cup U}((\omega_{S\setminus U}, \omega'_U), A) \\
&= \int\int K_{S\cup U}((\omega_{S\setminus U}, \omega'_{S\cap U}, \omega'_{U\setminus S}), A)\delta_{\omega_{S\cap U}}(d\omega'_{S\cap U})\mathbb{Q}(d\omega'_{U\setminus S}) \\
&= \int K_{S\cup U}((\omega_{S\setminus U}, \omega_{S\cap U}, \omega'_{U\setminus S}), A)\mathbb{Q}(d\omega'_{U\setminus S}) \\
&= \int \mathbb{Q}(d\omega'_{U\setminus S})K_{S\cup U}((\omega_S, \omega'_{U\setminus S}), A),
\end{aligned}$$

as required.

□

Theorem D.2. Suppose we have a causal space $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T}E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$, and let $U \in \mathcal{P}(T)$.

(i) For any measure $\mathbb{Q}$ on $\mathcal{H}_U$ and any causal mechanism $\mathbb{L}$ on $(\Omega, \mathcal{H}_U, \mathbb{Q})$, the causal kernel $K_U^{\text{do}(U,\mathbb{Q},\mathbb{L})} = K_U$ is a version of $\mathbb{P}_{\mathcal{H}_U}^{\text{do}(U,\mathbb{Q})}$, which means that $\mathcal{H}_U$ is a global source σ-algebra of the intervened causal space $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(U,\mathbb{Q})}, \mathbb{K}^{\text{do}(U,\mathbb{Q},\mathbb{L})})$.

(ii) Suppose $V \in \mathcal{P}(T)$ with $V \subseteq U$. Suppose that the measure $\mathbb{Q}$ on $(\Omega, \mathcal{H}_U)$ factorises over $\mathcal{H}_V$ and $\mathcal{H}_{U\setminus V}$, i.e. for any $A \in \mathcal{H}_V$ and $B \in \mathcal{H}_{U\setminus V}$, $\mathbb{Q}(A \cap B) = \mathbb{Q}(A)\mathbb{Q}(B)$. Then after a hard intervention on $\mathcal{H}_U$ via $\mathbb{Q}$, the causal kernel $K_V^{\text{do}(U,\mathbb{Q})}$ is a version of $\mathbb{P}_V^{\text{do}(U,\mathbb{Q})}$, which means that $\mathcal{H}_V$ is a global source σ-algebra of the intervened causal space $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(U,\mathbb{Q})}, \mathbb{K}^{\text{do}(U,\mathbb{Q})})$.

Proof. Suppose that $f = \sum_{i=1}^m b_i 1_{B_i}$ is a $\mathcal{H}_U$-simple function, i.e. with $B_i \in \mathcal{H}_U$ for $i = 1, ..., m$. Then for any $B \in \mathcal{H}_U$,

$$\begin{aligned}
\int_B f(\omega)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega) &= \int_B \sum_{i=1}^m b_i 1_{B_i}(\omega)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega) \\
&= \sum_{i=1}^m b_i \mathbb{P}^{\text{do}(U,\mathbb{Q})}(B \cap B_i) \\
&= \sum_{i=1}^m b_i \int \mathbb{Q}(d\omega)K_U(\omega, B \cap B_i) \quad \text{by the definition of } \mathbb{P}^{\text{do}(U,\mathbb{Q})} \\
&= \sum_{i=1}^m b_i \int \mathbb{Q}(d\omega)1_{B\cap B_i}(\omega) \quad \text{by Axiom 2.2(ii)} \\
&= \int_B \sum_{i=1}^m b_i 1_{B_i}(\omega)\mathbb{Q}(d\omega) \\
&= \int_B f(\omega)\mathbb{Q}(d\omega).
\end{aligned}$$

Now, for any $\mathcal{H}_U$-measurable map $g : \Omega \to \mathbb{R}$, we can write it as a limit of an increasing sequence of positive $\mathcal{H}_U$-simple functions f_n (see Section A.1), so for any $B \in \mathcal{H}_U$,

$$\begin{aligned}
\int_B g(\omega)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega) &= \int_B \lim_{n\to\infty} f_n(\omega)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega) \\
&= \lim_{n\to\infty} \int_B f_n(\omega)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega) \quad \text{by the monotone convergence theorem}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int_B f_n(\omega)\mathbb{Q}(d\omega) && \text{by above} \\
&= \int_B \lim_{n\to\infty} f_n(\omega)\mathbb{Q}(d\omega) && \text{by the monotone convergence theorem} \\
&= \int_B g(\omega)\mathbb{Q}(d\omega).
\end{aligned}
$$

We use this fact in the proof of both parts of this theorem.

(i) First note that we indeed have $K_U^{\text{do}(U,\mathbb{Q},\mathbb{L})} = K_U$, by Remark C.1(a). For any $A \in \mathcal{H}$, the map $\omega \mapsto K_U(\omega, A)$ is $\mathcal{H}_U$-measurable, so for any $B \in \mathcal{H}_U$,

$$
\begin{aligned}
\int_B K_U(\omega, A)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega) &= \int_B K_U(\omega, A)\mathbb{Q}(d\omega) && \text{by above fact} \\
&= \int 1_B(\omega)K_U(\omega, A)\mathbb{Q}(d\omega) \\
&= \int K_U(\omega, A\cap B)\mathbb{Q}(d\omega) && \text{by Axiom 2.2(ii)} \\
&= \mathbb{P}^{\text{do}(U,\mathbb{Q})}(A\cap B) \\
&= \int 1_{A\cap B}(\omega)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega) \\
&= \int 1_B(\omega)1_A(\omega)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega) \\
&= \int_B 1_A(\omega)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega).
\end{aligned}
$$

So $K_U(\cdot, A) = K_U^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\cdot, A)$ is indeed a version of the conditional probability $\mathbb{P}_{\mathcal{H}_U}^{\text{do}(U,\mathbb{Q})}(A)$, which means that $\mathcal{H}_U$ is a global source of $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(U,\mathbb{Q})}, \mathbb{K}^{\text{do}(U,\mathbb{Q},\mathbb{L})})$.

(ii) For any $A \in \mathcal{H}$, the map $\omega \mapsto K_V^{\text{do}(U,\mathbb{Q})}(\omega, A)$ is $\mathcal{H}_V$-measurable and hence $\mathcal{H}_U$-measurable, so for any $B \in \mathcal{H}_V \subseteq \mathcal{H}_U$,

$$
\begin{aligned}
&\int_B K_V^{\text{do}(U,\mathbb{Q})}(\omega_V, A)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega_V) \\
&= \int_B K_V^{\text{do}(U,\mathbb{Q})}(\omega_V, A)\mathbb{Q}(d\omega_V) && \text{by above fact} \\
&= \int K_V^{\text{do}(U,\mathbb{Q})}(\omega_V, A\cap B)\mathbb{Q}(d\omega_V) && \text{by Axiom 2.2(ii)} \\
&= \int\int \mathbb{Q}(d\omega'_{U\setminus V})K_U((\omega_V, \omega'_{U\setminus V}), A\cap B)\mathbb{Q}(d\omega_V) \\
&= \int K_U(\omega_U, A\cap B)\mathbb{Q}(d\omega_U) \\
&= \int_B 1_A(\omega)\mathbb{P}^{\text{do}(U,\mathbb{Q})}(d\omega).
\end{aligned}
$$

where, in going from the third line to the fourth, we used Theorem C.3, and to go from the fourth line to the fifth, we used the hypothesis that $\mathbb{Q}$ factorises over $\mathcal{H}_V$ and $\mathcal{H}_{U\setminus V}$, meaning $\mathbb{Q}(d\omega_{U\setminus V})\mathbb{Q}(d\omega_V) = \mathbb{Q}(d\omega_U)$. So $K_V^{\text{do}(U,\mathbb{Q})}(\omega, A)$ is indeed a version of the conditional probability $\mathbb{P}_{\mathcal{H}_V}^{\text{do}(U,\mathbb{Q})}(A)$, which means that $\mathcal{H}_V$ is a global source of $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(U,\mathbb{Q})}, \mathbb{K}^{\text{do}(U,\mathbb{Q})})$.

□

Lemma C.4. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U \in \mathcal{P}(T)$, $\mathbb{Q}$ a probability measure on $(\Omega, \mathcal{H}_U)$ and $\mathbb{L} = \{L_V : V \in \mathcal{P}(U)\}$ a causal mechanism on $(\Omega, \mathcal{H}_U, \mathbb{Q})$. Suppose we intervene on $\mathcal{H}_U$ via $(\mathbb{Q}, \mathbb{L})$.

(i) For $A \in \mathcal{H}_U$ and $V \in \mathcal{P}(T)$ with $V \cap U = \emptyset$, $\mathcal{H}_V$ has no causal effect on A (c.f. Definition B.1(i)) in the intervention causal space $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(U,\mathbb{Q})}, \mathbb{K}^{\text{do}(U,\mathbb{Q},\mathbb{L})})$, i.e. events in the σ-algebra $\mathcal{H}_U$ on which intervention took place are not causally affected by σ-algebras outside $\mathcal{H}_U$.

(ii) Again, let $V \in \mathcal{P}(T)$ with $V \cap U = \emptyset$, and also let $A \in \mathcal{H}$ be any event. If, in the original causal space, $\mathcal{H}_V$ had no causal effect on A, then in the intervention causal space, $\mathcal{H}_V$ has no causal effect on A either.

(iii) Now let $V \in \mathcal{P}(T)$, $A \in \mathcal{H}$ any event and suppose that the intervention on $\mathcal{H}_U$ via $\mathbb{Q}$ is hard. Then if $\mathcal{H}_V$ had no causal effect on A in the original causal space, then $\mathcal{H}_V$ has no causal effect on A in the intervention causal space either.

Proof. (i) Take any $S \in \mathcal{P}(T)$. See that

$$\begin{aligned}
K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A) &= \int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U) K_{S\cup U}((\omega_{S\setminus U}, \omega'_U), A) \\
&= \int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U) 1_A(\omega'_U) \\
&= \int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U) K_{(S\setminus V)\cup U}((\omega_{(S\setminus V)\setminus U}, \omega'_U), A) \\
&= \int L_{(S\setminus V)\cap U}(\omega_{(S\setminus V)\cap U}, d\omega'_U) K_{(S\setminus V)\cup U}((\omega_{(S\setminus V)\setminus U}, \omega'_U), A) \\
&= K_{S\setminus V}^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A)
\end{aligned}$$

where, in going from the first line to the second and from the second line to the third, we used the fact that $A \in \mathcal{H}_U$, and in going from the third line to the fourth, we applied the fact that $(S \setminus V) \cap U = S \cap U$ since $V \cap U = \emptyset$. Since $S \in \mathcal{P}(T)$ was arbitrary, $\mathcal{H}_V$ has no causal effect on A in the intervention causal space.

(ii) Take any $S \in \mathcal{P}(T)$. See that

$$\begin{aligned}
K_S^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A) &= \int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U) K_{S\cup U}((\omega_{S\setminus U}, \omega'_U), A) \\
&= \int L_{S\cap U}(\omega_{S\cap U}, d\omega'_U) K_{(S\cup U)\setminus V}((\omega_{(S\setminus V)\setminus U}, \omega'_U), A) \\
&= \int L_{(S\setminus V)\cap U}(\omega_{(S\setminus V)\cap U}, d\omega'_U) K_{(S\setminus V)\cup U}((\omega_{(S\setminus V)\setminus U}, \omega'_U), A) \\
&= K_{S\setminus V}^{\text{do}(U,\mathbb{Q},\mathbb{L})}(\omega, A)
\end{aligned}$$

where, in going from the first line to the second, we used the fact that $\mathcal{H}_V$ has no causal effect on A in the original causal space, and in going from the second line to the third, we used $U \cap V = \emptyset$, which gives us $S \cap U = (S \setminus V) \cap U$ and $(S \cup U) \setminus V = (S \setminus V) \cup U$. Since $S \in \mathcal{P}(T)$ was arbitrary, $\mathcal{H}_V$ has no causal effect on A in the intervention causal space.

(iii) Take any $S \in \mathcal{P}(T)$. Apply Theorem C.3 to see that

$$\begin{aligned}
K_S^{\text{do}(U,\mathbb{Q},\text{hard})}(\omega, A) &= \int \mathbb{Q}(d\omega'_{U\setminus S}) K_{S\cup U}((\omega_S, \omega'_{U\setminus S}), A) \\
&= \int \mathbb{Q}(d\omega'_{U\setminus S}) K_{(S\cup U)\setminus V}((\omega_S, \omega'_{U\setminus S}), A) && \text{Def. B.1(i)} \\
&= \int \mathbb{Q}(d\omega'_{U\setminus S}) K_{((S\setminus V)\cup U)\setminus V}((\omega_S, \omega'_{U\setminus S}), A) \\
&= \int \mathbb{Q}(d\omega'_{U\setminus S}) K_{(S\setminus V)\cup U}((\omega_S, \omega'_{U\setminus S}), A) && \text{Def. B.1(i)} \\
&= \int \mathbb{Q}(d\omega'_{U\setminus(S\setminus V)}) K_{(S\setminus V)\cup U}((\omega_{S\setminus V}, \omega'_{U\setminus(S\setminus V)}), A)
\end{aligned}$$

$$= K_{S\setminus V}^{\text{do}(U,\mathbb{Q})}(\omega, A),$$

where, in going from the second line to the third, we used that $(S\cup U)\setminus V = ((S\setminus V)\cup U)\setminus V$. Since $S \in \mathcal{P}(T)$ was arbitrary, $\mathcal{H}_V$ has no causal effect on A in the intervention causal space.

□

Lemma C.5. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U \in \mathcal{P}(T)$. For an event $A \in \mathcal{H}$, if $\mathcal{H}_U$ has a dormant causal effect on A in the original causal space, then there exists a hard intervention and a subset $V \subseteq U$ such that in the intervention causal space, $\mathcal{H}_V$ has an active causal effect on A.

Proof. That $\mathcal{H}_U$ has a dormant causal effect on A tells us that $K_U(\omega, A) = \mathbb{P}(A)$ for all $\omega \in \Omega$, but there exists some $S \in \mathcal{P}(T)$ and some $\omega_0 \in \Omega$ such that $K_S(\omega_0, A) \neq K_{S\setminus U}(\omega_0, A)$. We must have $S \cap U \neq \emptyset$, since otherwise $S \setminus U = S$ and we cannot possibly have $K_S(\omega_0, A) \neq K_{S\setminus U}(\omega_0, A)$. Then we hard-intervene on $\mathcal{H}_{S\setminus U}$ with the Dirac measure on ω_0. Then apply Theorem C.3 to see that

$$\begin{aligned} K_{S\cap U}^{\text{do}(S\setminus U,\delta_{\omega_0},\text{hard})}((\omega_0)_{U\cap S}, A) &= \int \delta_{\omega_0}(d\omega'_{S\setminus U}) K_S(((\omega_0)_{U\cap S}, \omega'_{S\setminus U}), A) \\ &= K_S(\omega_0, A) \\ &\neq K_{S\setminus U}(\omega_0, A) \end{aligned}$$

Note that the intervention measure on A is equal to $K_{S\setminus U}(\omega_0, A)$:

$$\mathbb{P}^{\text{do}(S\setminus U,\delta_{\omega_0})}(A) = \int \delta_{\omega_0}(d\omega'_{S\setminus U}) K_{S\setminus U}(\omega', A) = K_{S\setminus U}(\omega_0, A).$$

Putting these together, we have

$$K_{S\cap U}^{\text{do}(S\setminus U,\delta_{\omega_0},\text{hard})}(\omega_0, A) \neq \mathbb{P}^{\text{do}(S\setminus U,\delta_{\omega_0})}(A),$$

i.e. in the intervention causal space $(\Omega, \mathcal{H}, \mathbb{P}^{\text{do}(S\setminus U,\delta_{\omega_0})}, K_{S\cap U}^{\text{do}(S\setminus U,\delta_{\omega_0},\text{hard})})$, $\mathcal{H}_{S\cap U}$ has an active causal effect on A. □

Lemma C.6. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, and $U, V \in \mathcal{P}(T)$. For an event $A \in \mathcal{H}$, suppose that $\mathcal{H}_U$ has no causal effect on A given $\mathcal{H}_V$ (see Definition B.4). Then after an intervention on $\mathcal{H}_V$ via any $(\mathbb{Q}, \mathbb{L})$, $\mathcal{H}_{U\setminus V}$ has no causal effect on A.

Proof. Take any probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{H}_V)$ and any causal mechanism $\mathbb{L}$ on $(\Omega, \mathcal{H}_V, \mathbb{Q})$. Then see that, for any $S \in \mathcal{P}(T)$ and all $\omega \in \Omega$,

$$\begin{aligned} K_S^{\text{do}(V,\mathbb{Q},\mathbb{L})}(\omega, A) &= \int L_{S\cap V}(\omega_{S\cap V}, d\omega'_V) K_{S\cup V}((\omega_{S\setminus V}, \omega'_V), A) \\ &= \int L_{S\cap V}(\omega_{S\cap V}, d\omega'_V) K_{(S\cup V)\setminus(U\setminus V)}((\omega_{S\setminus(U\cup V)}, \omega'_V), A) \\ &= \int L_{(S\setminus(U\setminus V))\cap V}(\omega_{(S\setminus(U\setminus V))\cap V}, d\omega'_V) K_{(S\setminus(U\setminus V))\cup V}((\omega_{S\setminus(U\cup V)}, \omega'_V), A) \\ &= K_{S\setminus(U\setminus V)}^{\text{do}(V,\mathbb{Q},\mathbb{L})}(\omega, A), \end{aligned}$$

where, in going from the first line to the second, we used the fact that $\mathcal{H}_U$ has no causal effect on A given $\mathcal{H}_V$, and in going from the second line to the third, we used identities $S\cap V = (S\setminus(U\setminus V))\cap V$ and $(S \cup V) \setminus (U \setminus V) = (S \setminus (U \setminus V)) \cup V$. Since $S \in \mathcal{P}(T)$ was arbitrary, we have that $\mathcal{H}_{U\setminus V}$ has no causal effect on A in the intervention causal space. □

Theorem C.7. Let $(\Omega, \mathcal{H}, \mathbb{P}, \mathbb{K}) = (\times_{t\in T} E_t, \otimes_{t\in T}\mathcal{E}_t, \mathbb{P}, \mathbb{K})$ be a causal space, where the index set T can be written as $T = W \times \tilde{T}$, with W representing time and $\mathbb{K}$ respecting time. Take any $U \in \mathcal{P}(T)$ and any probability measure $\mathbb{Q}$ on $\mathcal{H}_U$. Then the intervention causal mechanism $\mathbb{K}^{\text{do}(U,\mathbb{Q},\text{hard})}$ also respects time.

Proof. Take any $w_1, w_@ \in W$ with $w_1 < w_2$. Since $\mathbb{K}$ respects time, we have that $\mathcal{H}_{w_2 \times \tilde{T}}$ has no causal effect on $\mathcal{H}_{w_1 \times \tilde{T}}$ in the original causal space. To show that $\mathcal{H}_{w_2 \times \tilde{T}}$ has no causal effect on $\mathcal{H}_{w_1 \times \tilde{T}}$ after a hard intervention on $\mathcal{H}_U$ via $\mathbb{Q}$, take any $S \in \mathcal{P}(T)$ and any event $A \in \mathcal{H}_{w_1 \times \tilde{T}}$. Then using Theorem C.3,

$$
\begin{aligned}
&K_S^{\mathrm{do}(U,\mathbb{Q},\mathrm{hard})}(\omega, A) \\
&= \int \mathbb{Q}(d\omega') K_{S \cup U}((\omega_S, \omega'_{U \setminus S}), A) \\
&= \int \mathbb{Q}(d\omega') K_{(S \cup U) \setminus \mathcal{H}_{w_2 \times \tilde{T}}}((\omega_{S \setminus \mathcal{H}_{w_2 \times \tilde{T}}}, \omega'_{U \setminus (S \cup \mathcal{H}_{w_2 \times \tilde{T}})}), A) \\
&= \int \mathbb{Q}(d\omega') K_{((S \cup U) \setminus \mathcal{H}_{w_2 \times \tilde{T}}) \cup (U \cap \mathcal{H}_{w_2 \times \tilde{T}})}((\omega_{S \setminus \mathcal{H}_{w_2 \times \tilde{T}}}, \omega'_{(U \setminus (S \cup \mathcal{H}_{w_2 \times \tilde{T}})) \cup (U \cap \mathcal{H}_{w_2 \times \tilde{T}})}), A) \\
&= \int \mathbb{Q}(d\omega') K_{(S \setminus \mathcal{H}_{w_2 \times \tilde{T}}) \cup U}((\omega_{S \setminus \mathcal{H}_{w_2 \times \tilde{T}}}, \omega'_{U \setminus (S \setminus \mathcal{H}_{w_2 \times \tilde{T}})}), A) \\
&= K_{S \setminus \mathcal{H}_{w_2 \times \tilde{T}}}^{\mathrm{do}(U,\mathbb{Q},\mathrm{hard})}(\omega, A)
\end{aligned}
$$

where, from the second line to the third, we used the fact that $\mathcal{H}_{w_2 \times \tilde{T}}$ has no causal effect on A, from the third line to the fourth we used the fact that $U \cap \mathcal{H}_{w_2 \times \tilde{T}}$ has no causal effect on A (by Remark B.2(e)) and Remark B.2(g), and from the fourth line to the fifth, we used that $((S \cup U) \setminus \mathcal{H}_{w_2 \times \tilde{T}}) \cup (U \cap \mathcal{H}_{w_2 \times \tilde{T}}) = (S \setminus \mathcal{H}_{w_2 \times \tilde{T}}) \cup U$ and $(U \setminus (S \cup \mathcal{H}_{w_2 \times \tilde{T}})) \cup (U \cap \mathcal{H}_{w_2 \times \tilde{T}}) = U \setminus (S \setminus \mathcal{H}_{w_2 \times \tilde{T}})$. Since $S \in \mathcal{P}(T)$ was arbitrary, we have that $\mathcal{H}_{w_2 \times \tilde{T}}$ has no causal effect on A (Definition B.1(i)). Since $A \in \mathcal{H}_{w_1 \times \tilde{T}}$ was arbitrary, $\mathcal{H}_{w_2 \times \tilde{T}}$ has no causal effect on $\mathcal{H}_{w_1 \times \tilde{T}}$, and so $\mathbb{K}^{\mathrm{do}(U,\mathbb{Q},\mathrm{hard})}$ respects time. $\square$