Slack-Free Spiking Neural Network Formulation for Hypergraph Minimum Vertex Cover Supplementary Material

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1 Derivation of Problem 1

Recall the QUBO for MVC

$$\min_{\mathbf{z}\in\{0,1\}^N} \quad \|\mathbf{z}\|_2^2 + \lambda \sum_{k=1}^{K} (1 - z_{i^{(k)}})(1 - z_{j^{(k)}}). \tag{1}$$

We expand

$$\begin{aligned}
&\|\mathbf{z}\|_2^2 + \lambda \sum_{k=1}^{K} (1 - z_{i^{(k)}})(1 - z_{j^{(k)}}) && (2)\\
=&\sum_{i=1}^{N} z_i^2 - \lambda \sum_{k=1}^{K} z_{i^{(k)}} - \lambda \sum_{k=1}^{K} z_{j^{(k)}} + \lambda \sum_{k=1}^{K} z_{i^{(k)}}.z_{j^{(k)}} + \text{constant} && (3)\\
=&\sum_{i=1}^{N} z_i^2 - \lambda \sum_{k=1}^{K} z_{i^{(k)}}^2 - \lambda \sum_{k=1}^{K} z_{j^{(k)}}^2 + \lambda \sum_{k=1}^{K} z_{i^{(k)}}.z_{j^{(k)}} + \text{constant} && (4)\\
=&\sum_{i=1}^{N} \left(1 - \lambda.\text{deg}(i)\right).z_i^2 + \lambda \sum_{k=1}^{K} z_{i^{(k)}}.z_{j^{(k)}} + \text{constant} && (5)\\
=&\mathbf{z}^T\mathbf{Q}\mathbf{z} + \text{constant} && (6)
\end{aligned}$$

where,

- Eq. (4) is obtained due to $z_i \in \{0,1\}$, thus $z_i = z_i^2$.
- In Eq. (5), $\text{deg}(\cdot)$ denotes the degree of i-th vertex.
- In Eq. (6), $\mathbf{Q}_{i,i} = (1 - \lambda.\text{deg}(i))$ for all $i = 1, \dots, N$, and $\mathbf{Q}_{i^{(k)},j^{(k)}} = \lambda$ for all $k = 1, \dots, K$

2 Derivation of Problem 2

Recall the QUBO for HMVC

$$\min_{\mathbf{z}\in\{0,1\}^N, \{\mathbf{y}^{(k)}\}_{k=1}^{K}\in\{0,1\}^{r'\times K}} \quad \|\mathbf{z}\|_2^2 + \lambda \sum_{k=1}^{K} \left(\mathbf{b}^{(k)^T}\mathbf{z} - \mathbf{1}_{r'}^T\mathbf{y}^{(k)} - 1\right)^2, \tag{7}$$

We define $\mathbf{x} = \begin{bmatrix} \mathbf{z} \\ \mathbf{y}^{(1)} \\ \vdots \\ \mathbf{y}^{(K)} \end{bmatrix} \in \{0,1\}^{N+r'K}$

First, as z_i is a binary variable, thus $z_i = z_i^2$, the first term of Eq. (7) can be rewritten

$$\|\mathbf{z}\|_2^2 = \sum_{i=1}^{N} z_i = \begin{bmatrix}\mathbf{x}^T & 1\end{bmatrix} .\mathbf{J}. \begin{bmatrix}\mathbf{x} \\ 1\end{bmatrix} \tag{8}$$

with $\mathbf{J} = \begin{bmatrix} \mathbf{I}_N & \mathbf{0}_{N\times(r'K+1)} \\ \mathbf{0}_{(r'K+1)\times N} & \mathbf{0}_{(r'K+1)\times(r'K+1)} \end{bmatrix}$, where $\mathbf{0}$ is a zero matrix with the size specified in its subscript, and $\mathbf{I}_N$ is the $N \times N$ identity matrix.

Next, we rewrite the second term of Eq. (7) as

$$\sum_{k=1}^{K} \left(\mathbf{b}^{(k)^T}\mathbf{z} - \mathbf{1}_{r'}^T\mathbf{y}^{(k)} - 1\right)^2 = \begin{bmatrix}\mathbf{x}^T & 1\end{bmatrix} \mathbf{H}^T\mathbf{H} \begin{bmatrix}\mathbf{x} \\ 1\end{bmatrix}, \tag{9}$$

with $\mathbf{H} = \begin{bmatrix}\mathbf{A}^T & -\mathbf{S} & -\mathbf{1}_K\end{bmatrix} \in \{0,1\}^{K\times(N+r'K+1)}$, where $\mathbf{A} = \begin{bmatrix}\mathbf{b}^{(1)} & \cdots & \mathbf{b}^{(K)}\end{bmatrix} \in \{0,1\}^{N\times K}$, $\mathbf{S} = \mathbf{I}_K \otimes \mathbf{1}_{r'}^T$ is the Kronecker product, $\mathbf{I}_K$ is a $K \times K$ identity matrix, and $\mathbf{1}$ is an all-ones vector with the size specified in its subscript.

Combining Eqs. (8) and (9), the optimization problem (7) can be rewritten as

$$\min_{\mathbf{x}\in\{0,1\}^{N+r'K}} \begin{bmatrix}\mathbf{x}^T & 1\end{bmatrix} \mathbf{Q} \begin{bmatrix}\mathbf{x} \\ 1\end{bmatrix}, \tag{10}$$

where $\mathbf{Q} = \mathbf{J} + \lambda\mathbf{H}^T\mathbf{H}$.

However, the problem (10) still has the constant 1 in the the variable, which can be removed using the following transformation

Suppose

$$\mathbf{x} = [x_1 \quad x_2 \quad x_3] \tag{11}$$

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ 0 & q_{22} & q_{23} & q_{24} \\ 0 & 0 & q_{33} & q_{34} \\ 0 & 0 & 0 & q_{44} \end{bmatrix}. \tag{12}$$

We then take the last column of $\mathbf{Q}$ except the last element q_{44}, which yields $\mathbf{q} = [q_{14} \quad q_{24} \quad q_{34}]^T$. Next, we add $\mathbf{q}$ to the diagonal of $\mathbf{Q}$

$$\mathbf{Q}' = \begin{bmatrix} q_{11}+q_{14} & q_{12} & q_{13} \\ 0 & q_{22}+q_{24} & q_{23} \\ 0 & 0 & q_{33}+q_{34} \end{bmatrix}. \tag{13}$$

Since $\mathbf{x}$ is a binary vector, i.e., $x_i^2 = x_i$, we can obtain

$$\begin{bmatrix}\mathbf{x}^T & 1\end{bmatrix} \mathbf{Q} \begin{bmatrix}\mathbf{x} \\ 1\end{bmatrix} = \mathbf{x}^T\mathbf{Q}'\mathbf{x} + q_{44}. \tag{14}$$

Apply the transformation described from Eq. (11) to Eq. (14) into the problem (10), we can obtain the standard QUBO form

$$\min_{\mathbf{x}\in\{0,1\}^{N+r'K}} \mathbf{x}^T\mathbf{Q}'\mathbf{x} + \text{constant}, \tag{15}$$