Universal In-Context Approximation By Prompting Fully Recurrent Models

Aleksandar Petrov, Tom A. Lamb, Alasdair Paren, Philip H.S. Torr, Adel Bibi
Department of Engineering Science
University of Oxford
aleks@robots.ox.ac.uk

Abstract

Zero-shot and in-context learning enable solving tasks without model fine-tuning, making them essential for developing generative model solutions. Therefore, it is crucial to understand whether a pretrained model can be prompted to approximate any function, i.e., whether it is a universal in-context approximator. While it was recently shown that transformer models do possess this property, these results rely on their attention mechanism. Hence, these findings do not apply to fully recurrent architectures like RNNs, LSTMs, and the increasingly popular SSMs. We demonstrate that RNNs, LSTMs, GRUs, Linear RNNs, and linear gated architectures such as Mamba and Hawk/Griffin can also serve as universal in-context approximators. To streamline our argument, we introduce a programming language called LSRL that compiles to these fully recurrent architectures. LSRL may be of independent interest for further studies of fully recurrent models, such as constructing interpretability benchmarks. We also study the role of multiplicative gating and observe that architectures incorporating such gating (e.g., LSTMs, GRUs, Hawk/Griffin) can implement certain operations more stably, making them more viable candidates for practical in-context universal approximation.

1 Introduction

Until recently, solving a task with machine learning required training or fine-tuning a model on a dataset matching the task at hand. However, large foundation models exhibit the ability to solve new tasks without being specifically fine-tuned or trained for them: often it is sufficient to simply prompt them in the right way. Prompting has been especially successful because of *in-context learning*: the ability to modify the model's behavior with information provided within the input sequence, without changing the underlying model parameters (Brown et al., 2020). Yet, we know little about the theoretical properties of prompting. It is not even clear if there are limits to what can be achieved with prompting or, conversely, whether it is possible to prompt your way into any behaviour or task.

This can be framed as a universal approximation question. Classically, universal approximation results show how a class of tractable functions, such as neural networks, approximates another class of concept functions, e.g., all continuous functions on a bounded domain, with arbitrary accuracy. This is often done by showing that one can choose *model parameters* that approximate the target function. However, in-context learning poses a different challenge as the model parameters are *fixed*. Instead, a part of the input (the prompt) is modified to cause the model to approximate the target function. Hence, we define universal *in-context* approximation to be the property that there exist fixed weights such that the resulting model can be prompted to approximate any function from a concept class. Understanding whether a model can be a universal *in-context* approximator is especially important as most commercial models are accessible exclusively via a prompting interface (La Malfa et al., 2023).

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In-context learning has been almost exclusively studied in conjunction with the transformer architecture (Vaswani et al., 2017). This is likely because in-context abilities appear once the models are large enough (Wei et al., 2021) and most large models have been transformer-based. On the subject of universal in-context approximation, Wang et al. (2023) were first to show that a transformer possesses this property by discretising and memorising all possible functions in the model weights. Memorisation is not needed, though, and even small transformers can be universal approximators when prompted Petrov et al. (2024). Both results, however, critically depend on the attention mechanism of the transformer architecture (Bahdanau et al., 2015).

Still, generative models are not restricted to attention-based architectures: there are the "classic" recurrent neural networks (RNNs, Amari, 1972), long short-term memory models (LSTMs, Hochreiter and Schmidhuber, 1997) and gated recurrent units (GRUs, Cho et al., 2014). Recently, Linear RNN models (also known as state-space models or SSMs) were proposed as a scalable alternative to the transformer architecture (Orvieto et al., 2023; Fu et al., 2023a) and have started to outperform similarly-sized transformers when multiplicative gating is added (Gu and Dao, 2023; De et al., 2024; Botev et al., 2024). Furthermore, despite in-context learning being associated with the transformer, recent empirical results show in-context learning in SSMs, RNNs, LSTMs and even convolutional models (Xie et al., 2022; Akyürek et al., 2024; Lee et al., 2024).

Yet, despite their ability to be in-context learners, there is little known about the theoretical properties of these fully recurrent architectures. As these architectures become more and more widely used, understanding their in-context approximation abilities is increasingly more important for their safety, security and alignment. We show that, in fact, many of these architectures, similarly to transformers, can be universal in-context approximators. Concretely, our contributions are as follows:

i. We develop *Linear State Recurrent Language* (LSRL): a programming language that compiles to different fully recurrent models. Programming in LSRL is akin to "thinking like a recurrent model". LSRL programs can then be implemented exactly as model weights.
ii. Using LSRL, we construct Linear RNN models that can be prompted to act as any token-to-token function over finite token sequences, or to approximate any continuous function. These results also hold for RNNs, LSTMs, GRUs and Hawk/Griffin models (De et al., 2024).
iii. We present constructions with and without multiplicative gating. However, we observe that the constructions without these gates depend on numerically unstable conditional logic.
iv. Nevertheless, we show that multiplicative gates lead to more compact and numerically stable models, making it more likely that universal in-context approximation properties arise in models utilising them, such as LSTMs, GRUs and the latest generation of Linear RNNs.

2 Preliminaries

Fully recurrent architectures. In this work, we focus exclusively on fully recurrent neural network architectures. Recurrent models operate over sequences. Concretely, consider an input sequence $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ with $\boldsymbol{x}_t \in \mathcal{X}$, $\mathcal{X}$ being some input space. We will refer to the elements of the input sequence as *tokens* even if they are real-valued vectors. A recurrent model $g : \mathcal{X}^\star \to \mathcal{Y}$ maps a sequence of inputs to an output in some output space $\mathcal{Y}$. These models are always causal, namely:

$$\boldsymbol{y}_t = g(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t). \tag{1}$$

We will abuse the notation and refer to $(\boldsymbol{y}_1, ..., \boldsymbol{y}_t)$=$(g(\boldsymbol{x}_1), ..., g(\boldsymbol{x}_1, ..., \boldsymbol{x}_t))$ as simply $g(\boldsymbol{x}_1, ..., \boldsymbol{x}_t)$. We will also separate the input sequence into a query $(\boldsymbol{q}_1, ..., \boldsymbol{q}_n)$ and a prompt $(\boldsymbol{p}_1, ..., \boldsymbol{p}_N)$. The prompt specifies the target function f that we approximate while the query designates the input at which we evaluate it. Contrary to the typical setting, we will place the query before the prompt.[1]

There are various neural network architectures that fall under the general framework of Eq. (1). The quintessential one is the RNN. It processes inputs one by one with only a non-linear state being passed from one time step to the other. A model g can thus be stacked RNN layers, each one being:

$$\begin{aligned} \boldsymbol{s}_t &= \sigma(\boldsymbol{A}\boldsymbol{s}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{b}), \\ \boldsymbol{y}_t &= \phi(\boldsymbol{s}_t), \end{aligned} \qquad \text{(Classic RNN)} \tag{2}$$

[1] That is necessitated by the limited capacity of the state variables. As the model is fixed, in order to increase the precision of the approximation, we can only increase the prompt length. If the prompt is before the query, it would have to be compressed into a fixed-size state, limiting the approximation precision even with increased prompt lengths. But if the query has a fixed size, it can be stored in a fixed-size state variable exactly.

with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{b}$ and the initial state value $\boldsymbol{s}_0$ being model parameters, σ a non-linear activation function and ϕ a multi-layer perceptron (MLP) with `ReLU` activations. We assume that σ is always a `ReLU` to keep the analysis simpler. The non-linearity in the state update can make the model difficult to train (vanishing and exploding gradients, Bengio et al., 1994). Therefore, Linear RNNs have been proposed as regularizing the eigenvalues of $\boldsymbol{A}$ can stabilise the training dynamics (Orvieto et al., 2023). Linear RNNs also admit a convolutional representation, making them trainable in parallel (Gu et al., 2021; Fu et al., 2023a). Linear RNNs drop the non-linearity from the state update in Eq. (2):

$$\begin{aligned} \boldsymbol{s}_t &= \boldsymbol{A}\boldsymbol{s}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{b}, \\ \boldsymbol{y}_t &= \phi(\boldsymbol{s}_t). \end{aligned} \qquad \text{(Linear RNN)} \qquad (3)$$

The fully linear state updates do not affect the expressivity of the models, as non-linear activations are nevertheless present in the MLP layers ϕ between the linear state update layers (Wang and Xue, 2023; Boyd and Chua, 1985). The state-of-the-art Linear RNN models also utilise some form of multiplicative gating (Gu and Dao, 2023; De et al., 2024; Botev et al., 2024). While specific implementations can differ, we can abstract it as the following Gated Linear RNN architecture:

$$\begin{aligned} \boldsymbol{s}_t &= \boldsymbol{A}\boldsymbol{s}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{b}, \\ \boldsymbol{y}_t &= \gamma(\boldsymbol{x}_t) \odot \phi(\boldsymbol{s}_t), \end{aligned} \qquad \text{(Gated Linear RNN)} \qquad (4)$$

with γ being another MLP and $\odot$ being the element-wise multiplication operation (Hadamard product). Eq. (4) encompasses a range of recently proposed models. For example, one can show that any model consisting of L stacked Gated Linear RNN layers, with γ and ϕ with k layers, can be represented as a $L(k{+}2)$-layer Hawk or Griffin model (De et al., 2024). The conversions are described in detail in App. E. We can similarly add multiplicative gating to the classic RNN architecture:

$$\begin{aligned} \boldsymbol{s}_t &= \sigma(\boldsymbol{A}\boldsymbol{s}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{b}), \\ \boldsymbol{y}_t &= \gamma(\boldsymbol{x}_t) \odot \phi(\boldsymbol{s}_t), \end{aligned} \qquad \text{(Gated RNN)} \qquad (5)$$

Eq. (5) may appear unusual but it is related to the well-known GRU (Cho et al., 2014) and LSTM (Hochreiter and Schmidhuber, 1997) architectures. Same as the case with Griffin/Hawk, any Gated RNN can be represented as a $L(k{+}2)$-layer GRU or LSTM model (details in Apps. C and D). As a result, if there exists a Gated RNN model that is a universal in-context approximator (which we later show to be the case), then there also exist GRU and LSTM models with the same property.

Theoretical understanding of in-context learning. Beyond the question of universal in-context approximation, there have been attempts to theoretically understand in-context learning from various perspectives. The ability to learn linear functions and perform optimization in-context has been extensively explored in the context of linear regression (Garg et al., 2022; Akyürek et al., 2022; von Oswald et al., 2023a; Fu et al., 2023b; Zhang et al., 2023; Ahn et al., 2023), kernel regression (Han et al., 2023) and dynamical systems (Li et al., 2023). Furthermore, studies have explored how in-context learning identifies and applies the appropriate pretraining skill (Xie et al., 2022; Coda-Forno et al., 2023; Bai et al., 2023). It has also been shown that transformers can construct internal learning objectives and optimize them during the forward pass (von Oswald et al., 2023b; Dai et al., 2023). However, these studies almost exclusively focus on the transformer architecture, and the applicability of their findings to fully recurrent models remains unclear.

Approximation theory. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces. Take a set of functions $\mathcal{C} \subseteq \mathcal{Y}^{\mathcal{X}}$ from $\mathcal{X}$ to $\mathcal{Y}$ called a *concept space*. Take also a set of nicely behaved functions $\mathcal{H} \subset \mathcal{Y}^{\mathcal{X}}$, called *hypothesis space*. $\mathcal{H}$ could be any set that we have tools to construct and analyse, e.g., all polynomials or all neural networks of a particular architectural type. Approximation theory is concerned with how well functions in $\mathcal{H}$ approximate functions in $\mathcal{C}$. We say that $\mathcal{H}$ *universally approximates* $\mathcal{C}$ over a compact domain $\mathcal{D}$ (or that $\mathcal{H}$ *is dense in* $\mathcal{C}$) if for every $f{\in}\mathcal{C}$ and $\epsilon{>}0$ there exist a $h{\in}\mathcal{H}$ such that $\sup_{\boldsymbol{x}\in\mathcal{D}} |f(\boldsymbol{x}){-}h(\boldsymbol{x})| {\le} \epsilon$. There is a long history of studying the concept class of continuous functions and hypothesis classes of single hidden layer neural networks (Cybenko, 1989; Barron, 1993) or deeper models (Hornik et al., 1989; Telgarsky, 2015). The concept class of sequence-to-sequence functions has been shown to be universally approximated with the hypothesis classes of transformers (Yun et al., 2019), RNNs (Schäfer and Zimmermann, 2006) and Linear RNNs (Wang and Xue, 2023).

The hypothesis spaces in this work are different. The model is fixed and only the prompt part of the input is changed, i.e., all learnable parameters are in the prompt. Take a recurrent model g as in Eq. (1) with *fixed* model parameters and a query length n. The hypothesis class is all functions that result by calling g on the user query followed by the prompt and taking the last n' outputs:

$$\mathcal{H}_g^{\mathcal{D}^n} = \{(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \mapsto g(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)[\text{-}n'{:}] \mid \forall \boldsymbol{p}_i \in \mathcal{D}, N > 0\}. \qquad (6)$$

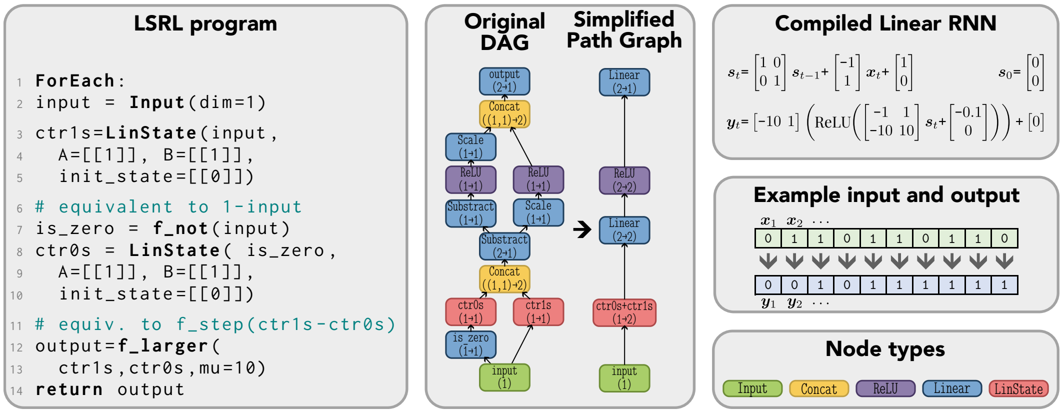

Figure 1: **Compilation of an LSRL program to a Linear RNN.** An example of a simple LSRL program that takes a sequence of 0s and 1s as an input and outputs 1 if there have been more 1s than 0s and 0 otherwise. The LSRL compiler follows the rules in App. A to simplify the computation DAG into a path graph. The resulting path graph can be represented as a Linear RNN with one layer.

The domain $\mathcal{D}$ of $\boldsymbol{p}_i$ and $\boldsymbol{q}_i$ can be continuous embeddings in $\mathbb{R}^d$ or discrete tokens $\mathcal{V} = \{1, ..., V\}$.

Note that each $h{\in}\mathcal{H}_g$ is identified by a prompt $(\boldsymbol{p}_1, ..., \boldsymbol{p}_N)$ but is a function with domain all possible queries $(\boldsymbol{q}_1, ..., \boldsymbol{q}_n)$. Therefore, finding a hypothesis $h{\in}\mathcal{H}_g$ that approximates a target function f is equivalent to finding the prompt of that hypothesis. The approximation properties of $\mathcal{H}_g$ in Eq. (6) depend on the architecture of g, as well as its specific parameters.

We study the recurrent architectures in Eqs. (2) to (5) and their ability to approximate continuous functions over real-valued vectors and to represent discrete maps over tokens (which corresponds to how language models are used in practice). We consider the following classes of functions. $\mathcal{C}^{\text{vec}}{=}(\mathbb{R}^{d_{\text{out}}})^{[0,1]^{d_{\text{in}}}}$ contains all continuous functions from the unit hypercube to $\mathbb{R}^{d_{\text{out}}}$, while $\mathcal{C}^{\text{tok}}{=}\{h{\in}(\mathcal{V}^l)^{\mathcal{V}^l} \mid h \text{ causal}\}$ all causal functions from l tokens to l tokens. The hypothesis classes are $\mathcal{H}^{\text{vec}}(g)$ corresponding to Eq. (6) with $D{=}[0,1]^{d_{\text{in}}}$, $n{=}n'{=}1$ and g some *fixed* model of one of the four architectures in Eqs. (2) to (5), and $\mathcal{H}^{\text{tok}}(g)$ with $D{=}\mathcal{V}$ and $n{=}n'{=}l$.

3 Linear State Recurrent Language (LSRL)

We can construct the weights for universal in-context models with the architectures in Eqs. (2) to (5) by hand but this is labour-intensive, error-prone, difficult to interpret, and the specific weights would be architecture-dependent. Working at such a low level of abstraction can also obfuscate common mechanisms and design patterns, making it more difficult to appreciate both the capabilities and the constraints of fully recurrent architectures. Instead, we propose a new programming language: *Linear State Recurrent Language* (LSRL).[2] LSRL programs compile to the four architectures in Eqs. (2) to (5). Conversely, any Linear RNN can be represented as an LSRL program, making LSRL a versatile tool for studying the capabilities of recurrent models. Later, in Secs. 4 to 6 we make use of LSRL to develop programs that are universal approximators for $\mathcal{C}^{\text{vec}}$ and $\mathcal{C}^{\text{tok}}$, thus showing that all four architectures can be universal in-context approximators.

LSRL syntax. An LSRL program specifies how a single element is processed and how the recurrent states are updated for the next element. LSRL programs always start with an `Input`$(\boldsymbol{x}) = \boldsymbol{x}$ with an $\boldsymbol{x}$ of a fixed dimension. Only one `Input` can be declared in a program. Linear layers and ReLUs are also supported: `Lin`$[\boldsymbol{A}, \boldsymbol{b}](\boldsymbol{x}) := \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, `ReLU`$(\boldsymbol{x}) := \max(\boldsymbol{0}, \boldsymbol{x})$. The unique component of LSRL, however, is its `LinState` operation implementing the linear state update in Linear RNNs (Eq. (3)): `LinState`$[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{b}, \boldsymbol{s}_0](\boldsymbol{x}_t) := \boldsymbol{A}\boldsymbol{s}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{b}$, where the state $\boldsymbol{s}_{t-1}$ is the output of the call this node at step $t-1$. `LinState` is the only way information can be passed from previous tokens to the current one. We also provide a `Concat` operation that combines

[2]Our implementation of LSRL is available at `https://github.com/AleksandarPetrov/LSRL`

```
ForEach:
input = Input(dim=1+d_in+d_out)
# counter needed to know whether we are looking at the query or the prompt
const_1 = f_constant(input, 1)
counter_vector = LinState(input=const_1, A=ones(d_in,d_in), B=ones(d_in,1), init_state=zeros(d_in,1))
# copy the query in a state (only when the counter is 1)
q_update = f_ifelse(cond=f_smaller(counter_vector, 1.5), t=input[: d_in], f=input[: d_in]*0)
q = LinState(input=q_update, A=eye(d_in), B=eye(d_in), init_state=zeros(d_in,1))
# the following operations will only change the output when counter > 1
# the step size is the first element of every prompt element
step_size = Linear(input=input[0], A=ones(d_in,1), b=zeros(d_in,1))
# using it we can compute the upper bounds of the current prompt cell
lb = input[1 : 1 + d_in]
ub = lb + step_size
# now check if q is in this cell (the bump should be 1 on all dimensions)
q_in_bump_componentwise = f_bump(q, lb, ub)
bump_sum = Linear(input=q_in_bump_componentwise, A=ones(1,d_in), b=zeros(1,1))
in_cell = f_larger(bump_sum, d_in - 0.5)
in_and_processing = f_and(in_cell, f_larger(counter, 0.5))
# if counter>1 and this cell contains q, add the value to the output state
update = f_ifelse(cond=f_larger(in_and_processing,0.5), t=input[-d_out:], f=input[-d_out:]*0)
y = LinState(input=update, A=eye(d_out), B=eye(d_out), init_state=zeros(d_out,1))
return y
```

Listing 1: **LSRL program for universal approximation in-context for continuous functions.** The inputs are $\boldsymbol{q} = [\boldsymbol{q}'^\top, \boldsymbol{0}_{d_\text{out}+1}^\top]^\top$ with $\boldsymbol{q}' \in [0,1]^{d_\text{in}}$ being the query value at which we want to evaluate the function, then followed by prompts describing the target function as in Eq. (8).

variables: $\texttt{Concat}(\boldsymbol{x}, \boldsymbol{y}) := (\boldsymbol{x}_1, ..., \boldsymbol{x}_{|\boldsymbol{x}|}, \boldsymbol{y}_1, ..., \boldsymbol{y}_{|\boldsymbol{y}|})$. Finally, to support gating architectures we also implement a rudimentary `Multi` operation that splits its input into two sub-arrays and returns their element-wise multiplication: $\texttt{Multi}(\boldsymbol{x}) := \boldsymbol{x}[\,:\, {}^{|\boldsymbol{x}|}/_2] \odot \boldsymbol{x}[{}^{|\boldsymbol{x}|}/_2 :\,]$. Naturally, `Multi` requires that $\boldsymbol{x}$ has even length. These six operations can be composed into a direct acyclic graph (DAG) with a single source node (the `Input` variable) and a single sink node (marked with a `return` statement).

Such a program operates over a single token $\boldsymbol{x}_t$ passed to `Input`, while a recurrent model needs to operate over sequences. Thus, we wrap the program into a `ForEach` loop that passes each element individually for the DAG to output a variable denoted by a `return` clause. Each element is processed by the exact same program, with the only difference being that the state of the `LinState` variables is changing between iterations. You can see an example of a small LSRL program in Fig. 1.

Expressiveness limitations. `ForEach` does not behave like the typical `for` loop: only the states are accessible between iterations, i.e., you cannot use the output of a linear layer at step t in any computation at step $t+1$. Furthermore, as the program is a DAG and only states of `LinState` nodes are passed between iterations, variables computed in latter operations of a previous time step are not accessible as inputs in earlier layers (with respect to the topological sorting of the computation graph). This leads to a key programming paradigm in LSRL: a `LinState` update cannot depend non-linearly on its own state. That includes it depending on a variable that depends on the `LinState` itself and conditional updates to the state. Such a dependency would break the DAG property of the program.[3] This poses serious limitations on what algorithms can be expressed in a Linear RNN and makes programming them challenging. Still, in Sec. 4 we show how carefully constructing state updates and auxiliary variables can nevertheless allow to program some limited conditional behaviours.

Compilation. Any LSRL program without `Multi` nodes can be compiled to a Linear RNN (Eq. (3)) or to a Gated Linear RNN (Eq. (4)). If the program has `Multi` nodes, then it cannot be compiled to a Linear RNN as the multiplicative gating cannot be implemented exactly. However, it can be compiled to a Gated Linear RNN. To compile an LSRL program to a Linear (Gated) RNN, we first parse the program to build a computation graph. This is a DAG with a single source (the `Input` node) and a single sink (the `return` statement of the `ForEach` loop). At the same time, a Linear (Gated) RNN can be represented as a path graph (no branching) with the six basic operations as nodes. Therefore, the compilation step needs to transform this DAG into a path graph. We achieve that by iterativly collapsing the first branching point into a single node. The exact rules that achieve that are described in App. A. Later, in Sec. 6, we will show how any Linear (Gated) RNN can be converted into a *non-linear* (Gated) RNN, hence, how we can compile LSRL programs to these architectures as well.

[3] For example, we cannot implement an operation that adds one to the state and squares it at each time step: $s_{t+1} = (s_t + 1)^2$ or an operation that performs conditional assignment $s_{t+1} = 0 \texttt{ if } (s_t > 5) \texttt{ else } s_t$.

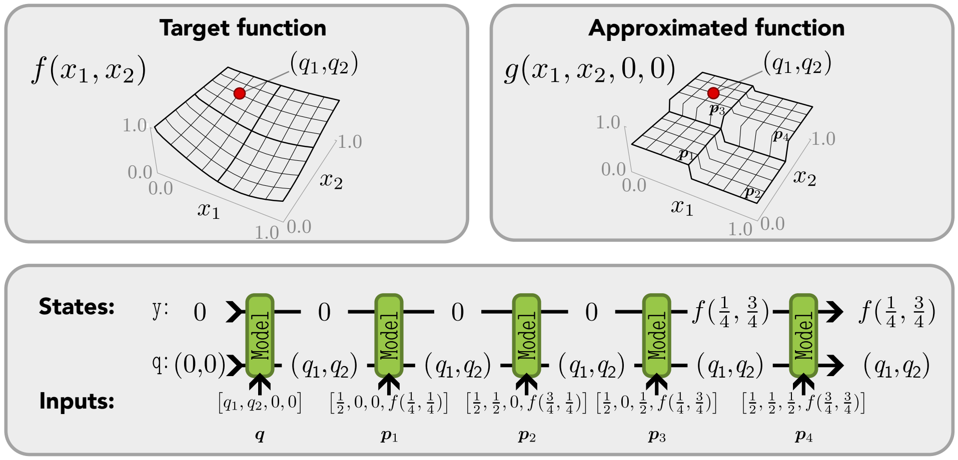

Figure 2: **Intuition behind the LSRL program for universal in-context approximation for continuous functions in Lst. 1.** Our target function f has input dimension $d_{\text{in}} = 2$ and output dimension $d_{\text{out}} = 1$. Each input dimension is split into two parts, hence $\delta = 1/2$. We illustrated an example input sequence of length 5: one for the query and four for the prompt tokens corresponding to each of the discretisation cells. The query (q_1, q_2) falls in the cell corresponding to the third prompt token. We show how the two `LinState` variables in the program are updated after each step. Most notably, how the state holding the output `y` is updated after $\boldsymbol{p}_3$ is processed.

Syntactic sugar. To make programming easier, we define several convenience functions. For instance, we can `Slice` variables $\boldsymbol{x}[l{:}u]$ via sparse `Lin` layers. We can also sum variables and element-wise multiplication with scalars (implemented as `Lin` layers). For logical operations we also need step functions which can be approximated with `ReLU`s: $\texttt{f\_step}[\mu](\boldsymbol{x}) := \texttt{ReLU}(\mu\boldsymbol{x}) - \mu\texttt{ReLU}(\boldsymbol{x} - 1/\mu)$, where μ is a positive constant controlling the quality of the approximation. We can also approximate bump functions (1 between l and u and 0 otherwise): $\texttt{f\_bump}[\boldsymbol{l}, \boldsymbol{u}, \mu](\boldsymbol{x}) := \texttt{f\_step}[\mu](\boldsymbol{x} - \boldsymbol{l}) - \texttt{f\_step}[\mu](\boldsymbol{x} - \boldsymbol{u})$. Similarly, we can approximate conjunction (`f_and`), disjunction (`f_or`), negation (`f_not`), and comparison operators (`f_larger` and `f_smaller`). See App. F for the definitions.

Critically, we need also a conditional operator that assigns a value $\texttt{t}(\boldsymbol{x})$ if a certain condition is met and another value $\texttt{f}(\boldsymbol{x})$ otherwise. One way to implement this is:

$$
\begin{aligned}
\texttt{f\_ifelse}[\texttt{cond}, \texttt{t}, \texttt{f}, \lambda](\boldsymbol{x}) := {} & \texttt{ReLU}(\texttt{-}\lambda\,\texttt{cond}(\boldsymbol{x})\texttt{+f}(\boldsymbol{x})) + \texttt{ReLU}(\texttt{-}\lambda\,\texttt{f\_not}(\texttt{cond}(\boldsymbol{x}))\texttt{+t}(\boldsymbol{x})) \\
& - \texttt{ReLU}(\texttt{-}\lambda\,\texttt{cond}(\boldsymbol{x})\texttt{-f}(\boldsymbol{x})) - \texttt{ReLU}(\texttt{-}\lambda\,\texttt{f\_not}(\texttt{cond}(\boldsymbol{x}))\texttt{-t}(\boldsymbol{x})),
\end{aligned} \tag{7}
$$

where λ is a constant that is larger than any absolute value that $\texttt{t}(\boldsymbol{x})$ and $\texttt{f}(\boldsymbol{x})$ can attain. This construction, however, is not numerically stable and we will study alternatives in Sec. 5. We provide both numerical (`SciPy.sparse`, Virtanen et al. 2020) and symbolic (`SymPy`, Meurer et al. 2017) backends with the second being crucial for programs that are not numerically stable.

Prior work on encoding algorithms in model weights. A similar approach to developing a programming language that compiles to model weights was already done for the transformer architecture with the RASP language (Weiss et al., 2021) and the Tracr compiler (Lindner et al., 2023). They were predominantly created as a tool for interpretability research. In a sense, RASP is to a transformer as LSRL is to a (Linear) (Gated) RNN. Hence, can be used to develop benchmarks for interpretability methods for fully-recurrent architectures. However, while RASP can only express a subset of transformer models, LSRL is isomorphic to the set of all (Gated) Linear RNNs (though not to the non-linear ones). That means that any (Gated) Linear RNN can be represented and analysed as an LSRL program and vice versa. Hence, the limitations of what you can express in LSRL are also limitations of what a Linear (Gated) RNN can do. Namely: (*i*) we cannot have exact multiplicative interactions between inputs without multiplicative gates, and (*ii*) we cannot have state variable updates depending non-linearly on their previous iterations or in any way on a variable that depends on them.

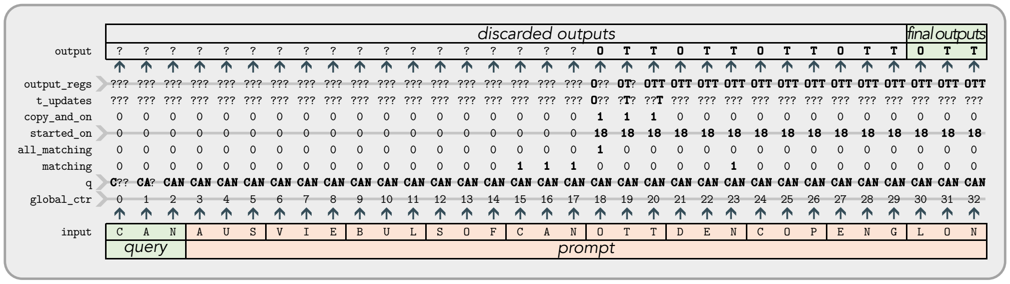

Figure 3: **Intuition behind the LSRL program for universal in-context approximation for discrete functions in Lst. 2.** Our keys and values have length n=3 and represent countries and capitals, e.g., `AUS`tria↦`VIE`nna, `BUL`garia↦`SOF`ia, and so on. The query is `CAN` for Canada and the final n outputs are `OTT` (Ottawa). We show the values of some of the variables in Lst. 2 at each step, with the `LinState` variables being marked with arrows. For cleaner presentation we are tokenizing letters as `0`↦`?`, `1`↦`A`, `2`↦`B`, etc. Vertical separators are for illustration purposes only.

4 Universal In-Context Approximation with Linear RNNs

We proceed with building LSRL programs that are universal in-context approximators: one for approximating continuous functions ($\mathcal{C}^{\text{vec}}$), and one for maps between token sequences ($\mathcal{C}^{\text{tok}}$).

4.1 Approximating continuous functions in $\mathcal{C}^{\text{vec}}$

The idea behind the approximation for continuous functions is to discretise the domain into a grid and approximate the function as constant in each cell of the grid. This technique is commonly used for showing universal approximation using the step activation function (Blum and Li, 1991; Scarselli and Tsoi, 1998). However, it is not obvious how to implement this approach in-context when information across input tokens can be only combined linearly. Consider a target function $f : [0,1]^{d_{\text{in}}} \to [0,1]^{d_{\text{out}}}$ and a discretization step δ. Our approach is to describe the value of f in each of the discretization cells as a single prompt token. For the cell with lower bounds $l_1, \ldots, l_{d_{\text{in}}}$ and their respective upper bounds l_1+δ, ..., $l_{d_{\text{in}}}$+δ, the corresponding prompt token is a (d_{in}+d_{out}+1)-dimensional vector:

$$\boldsymbol{p} = [\delta, l_1, \ldots, l_{d_{\text{in}}}, \bar{\boldsymbol{y}}_1, \ldots \bar{\boldsymbol{y}}_{d_{\text{out}}}]^\top, \tag{8}$$

where $\bar{\boldsymbol{y}}$ is the value of f at the centre of that cell: $\bar{\boldsymbol{y}} = f(l_1 + \delta/2, \ldots, l_{d_{\text{in}}} + \delta/2)$. Each prompt token describes the size of the cell (the discretisation step δ), its starting lower bound, and the value of the target function at the centre of the cell. Thus, $\lceil 1/\delta \rceil^{d_{\text{in}}}$ such tokens, one for each cell, are sufficient to describe the piece-wise constant approximation of f. A query $\boldsymbol{q}' \in [0,1]^{d_{\text{in}}}$ can fall in only one of the cells. We pad it with zeros and encode it as the first input element: $\boldsymbol{q} = [\boldsymbol{q}'^\top, \boldsymbol{0}^\top_{d_{\text{out}}+1}]^\top$, followed by the prompt. Our program will extract and save $\boldsymbol{q}'$ to a state and then process the prompt tokens one at a time until it finds the one whose cell contains $\boldsymbol{q}'$. The target function value for this cell will be added to an accumulator state. If the current cell does not contain $\boldsymbol{q}'$, then 0 is instead added.Hence, the accumulator's final value corresponds to the value of f at the centre of the cell containing $\boldsymbol{q}'$. The full LSRL program is provided in Lst. 1 and an illustration for $d_{\text{in}} = 2, d_{\text{out}} = 1, \delta = 1/2$ is shown in Fig. 2. The prompt length required to approximate an L-Lipschitz function f (w.r.t. the ℓ_2 norm) to precision ϵ is $N = \left(2\epsilon/L\sqrt{d_{\text{in}}}\right)^{-d_{\text{in}}} = \mathcal{O}(\epsilon^{-d_{\text{in}}})$ (see App. B for the proof). Asymptotically, this is as good as one can hope without further assumptions on the target function. This is also better than the best known result for the same problem for transformers: $\mathcal{O}(\epsilon^{-10-14d_{\text{in}}-4d_{\text{in}}^2})$ in Petrov et al. 2024.

4.2 Approximating functions over token sequences in $\mathcal{C}^{\text{tok}}$

Sec. 4.1 focused on continuous functions but recurrent architectures are often used to model natural language whose domain is tokens. Thus, we also look at modelling maps over a discrete domain. Any function from n tokens to n tokens taking values in $\mathcal{V} = \{1, \ldots, V\}$ can be represented as a dictionary whose keys and values are in $\mathcal{V}^n$. Therefore, a simple way to represent this function in-context is to first provide the n tokens corresponding to the query and then a sequence of $2n$ tokens corresponding to key and value pairs (see Fig. 3 for an illustration of the setup). The model stores the

```
1  ForEach:
2  input = Input(dim=1)
3  # counter needed to know whether we are looking at the query or the prompt
4  const_1 = f_constant(input, 1)
5  global_ctr = LinState(input=const_1, A=[[1]], B=[[1]], init_state=[[-1]])
6  # counters mod[n] and mod[2n]
7  mod_n_ctr = f_modulo_counter(input, n)
8  mod_2n_ctr = f_modulo_counter(input, 2*n)
9  # which mode are we in (looking at the query, comparing query with key, or copying value to state)
10 is_prompt = f_larger(global_ctr, n-0.5)
11 is_compare_mode = f_larger(mod_2n_ctr, n-0.5)
12 is_copy_mode = f_and(is_prompt, f_not(is_compare_mode))
13 is_first_token_for_copy = f_and(is_copy_mode,f_smaller(mod_n_ctr, 0.5))
14 # update the state holding the query if this is one of the first n tokens
15 tq=f_ifelse(f_smaller(is_prompt, 0.5), t=input, f=input*0)
16 tqs=[f_ifelse(f_and(f_larger(mod_n_ctr,i-0.5), f_smaller(mod_n_ctr,i+0.5)),t=tq,f=tq*0) for i in 1..n]
17 q = LinState(input=Concat(tqs), A=eye(n), B=eye(n), init_state=zeros(n,1)) # query
18 # if we are in compare mode (looking at keys), check if this token matches the corresponding one in the query
19 qs=[f_ifelse(f_and(f_larger(mod_n_ctr,i-0.5),f_smaller(mod_n_ctr,i+0.5)),t=q[i],f=q[i]*0) for i in 1..n]
20 cor_q_el = Linear(input=Concat(qs), A=ones(1,n), b=zeros(1,1))
21 matching = f_and(f_and(f_larger(input, cor_q_el-0.5),f_smaller(input, cor_q_el+0.5)),is_compare_mode)
22 # keep a buffer of the last last n+1 match values, the +1 because we can only read the buffer after writing
23 buffer = LinState(input=matching, A=shift_matrix, B=[[0] for _ in 1..n], [[1]]), init_state=zeros(n+1,1))
24 buffer_sum = Linear(input=buffer, A=[[1 for _ in 1..n], [0]], b=zeros(1, 1))
25 all_matching = f_larger(buffer_sum, n-0.5)
26 # if all are matching and it's the first token in the value part of the (key, value) pair, then mark this as
        the iterations when we start copying to state
27 matching_and_first_for_copy = f_and(all_matching, is_first_token_for_copy)
28 t_started_on_update = f_ifelse(matching_and_first_for_copy,t=global_ctr, f=global_ctr*0)
29 started_on = LinState(input=t_started_on_update, A=eye(1), B=eye(1), init_state=zeros(1, 1))
30 # copying to state for n iterations after started_on
31 copy_and_on = f_and(is_copy_mode, f_smaller(global_ctr, started_on+n))
32 mod_n_eq_i = [ f_and(f_larger(mod_n_ctr,i-0.5), f_smaller(mod_n_ctr,i+0.5)) for i in 1..n ]
33 t_updates_should_update = [f_and(copy_and_on, mod_n_eq_i[i]) for i in 1..n]
34 t_updates = [f_ifelse(f_larger(t_updates_should_update[i], 0.5), t=input, f=input*0) for i in 1..n]
35 output_regs = [LinState(input=update, A=eye(1), B=eye(1), init_state=zeros(1,1)) for update in t_updates]
36 # finally, read out the value from the corresponding output register in order to output from the model
37 t_outputs = [f_ifelse(f_larger(mod_n_eq_i[i], 0.5), t=output_regs[i], f=output_regs[i]*0) for i in 1..n]
38 return Linear(input=Concat(t_outputs), A=ones(1,n), b=zeros(1,1))
```

Listing 2: **LSRL program for universal in-context approximation of discrete functions.** The inputs are $\boldsymbol{q}_1, ..., \boldsymbol{q}_n$ (the query tokens), followed by pairs of keys and values from the map we are approximating. The last n outputs are the value corresponding to the key matching the query.

query in a state and processes the key-value pairs one by one by comparing the key (the first n tokens) with the query. If they match, then the value (the next n tokens) is copied into a state that keeps it and repeatedly outputs it. This continues until the end of the prompt, at which point the last n outputted tokens will be the value corresponding to the key matching the query. This is essentially a dictionary lookup. However, as shown in Lst. 2, implementing dictionary lookup in a linear recurrent model is much less straightforward than executing `dict[key]` in a general-purpose programming language.

Lst. 2 can appear daunting at first so we would like to clarify the non-trivial aspects. First, we need to count how far we are into every set of n or $2n$ tokens. This can be done with $\bmod n$ and $\bmod 2n$ operations but implementing modulo for arbitrary large inputs is not possible with `ReLU` MLPs (Ziyin et al., 2020). Therefore, we implement this with `LinState` as `f_modulo_counter` which has a unit-length state that is rotated $1/n$ or $1/2n$ revolutions per iteration, with the angle corresponding to the modulo value (App. F.7). Second, we need to do dynamic indexing to copy the i-th input in a subsequence to the i-th element of a state and vice-versa. Dynamic indexing, however, cannot be succinctly represented in a Linear RNN. We work around this with temporary variables that are non-zero only at the i-th coordinates (see Lines 16, 17, 19, 20, 32 to 35, 37 and 38). Finally, in order to compare whether all n elements in the query and the key match, we need to remember whether the previous n pairs were matching. As RNNs do not have attention, we implement this short-term memory buffer as a `LinState` with a shift matrix (Line 23).

5 Stable Universal In-Context Approximation with Gated Linear RNNs

The ReLU-based conditional operator is not numerically stable. The LSRL programs in Lsts. 1 and 2 for approximating functions in respectively $\mathcal{C}^{\text{vec}}$ and $\mathcal{C}^{\text{tok}}$ rely on the `f_ifelse` conditional assignment operator in Eq. (7) in order to implement different behaviours depending on whether

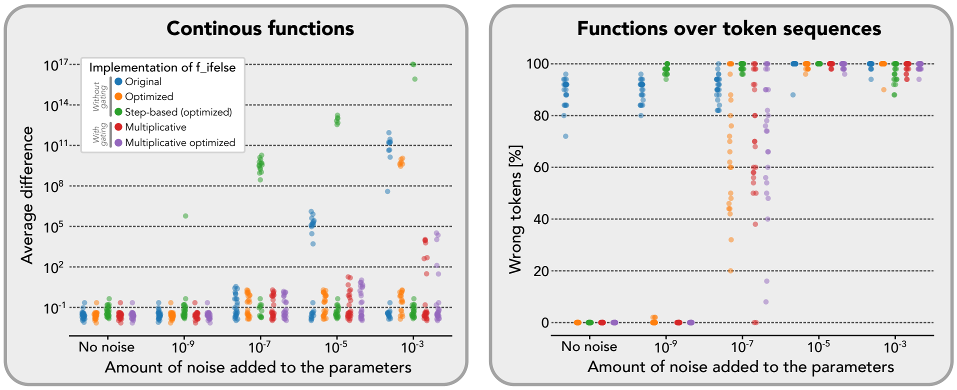

Figure 4: **Robustness of the various `f_ifelse` implementations to model parameter noise.** We show how the performance of the two universal approximation programs in Lsts. 1 and 2 deteriorates as we add Gaussian noise of various magnitudes to the non-zero weights of the resulting compiled models. As expected, the original `f_ifelse` implementation in Eq. (7) exhibits numerical precision errors at the lowest noise magnitude. For the token sequence case, numerical precision errors are present in all samples even in the no-noise setting. Hence, the original `f_ifelse` implementation is less numerically robust while the implementations with multiplicative gating are the most robust. For Lst. 1 (approximating $\mathcal{C}^{\text{vec}}$) we report the Euclidean distance between the target function value and the estimated one over 10 queries for 25 target functions. For Lst. 2 we report the percentage of wrong token predictions over 5 queries for 25 dictionary maps. Lower values are better in both cases.

we are processing the query or specific parts of the prompt. This operator is not numerically stable. The first term in Eq. (7) relies on $\texttt{cond}(\boldsymbol{x})$ being exactly zero if the condition is not met. In this way, multiplying it with $-\lambda$ would be 0 and $\texttt{f}(\boldsymbol{x})$ would be returned. However, if $\texttt{cond}(\boldsymbol{x})$ is not identically 0 but has a small positive value, then $-\lambda\texttt{cond}(\boldsymbol{x})$ can "overpower" $\texttt{f}(\boldsymbol{x})$ resulting in the `ReLU` output being 0. In our experience, this is not a problem when processing inputs through the LSRL program step-by-step. However, de-branching the DAG into a path graph —which is necessary in order to uncover the equivalent Linear RNN— appears to introduce such numerical instabilities which occasionally result in wrong outputs as conditional assignments will be 0 when they should not. This problem is more prominent in Lst. 2 which is longer (more debranching steps) and has more `f_ifelse` operations: it gets most tokens wrong because of that instability (see *Original, No noise* in Fig. 4). To this end, we support LSRL with a symbolic backend (based on SymPy) that performs the debranching steps exactly. Using it, both programs always produce the correct output.

This numerical instability highlights a critical practical limitations of the universal approximation results in Sec. 4: if the models are not numerically stable, it is unlikely that they occur in practice by training models using gradient descent. This section shows how to improve the numerical stability of Eq. (7) and obtain more realistic recurrent models that are universal approximators in-context.

Removing unnecessary terms in Eq. (7). Eq. (7) has 4 separate `ReLU` terms. The first two handle the cases when $\texttt{t}(\boldsymbol{x})$ and $\texttt{f}(\boldsymbol{x})$ are positive and the second two when they are negative. Therefore, if we know that one or both of these will always be non-negative, we can drop the corresponding terms. Additionally, if $\texttt{f}(\boldsymbol{x})$ is always 0, then the first and third terms can be safely dropped. Similarly, the second and fourth are unnecessary if $\texttt{f}(\boldsymbol{x}) \equiv 0$. All `f_ifelse` in Lsts. 1 and 2 fall in this case and hence can be simplified. We will refer to this `f_ifelse` implementation that is aware of the attainable values of $\texttt{t}(\boldsymbol{x})$ and $\texttt{f}(\boldsymbol{x})$ as `optimized`. As it reduces the number of numerically unstable `ReLU` operations in the model, we expect that it will improve the stability of the compiled models. We experimented with adding various levels of noise to the non-zero model parameters, and, as the results in Fig. 4 show, `optimized` is indeed more numerically robust than `original`.

Step-based implementation. We can get rid of the input sensitivity of Eq. (7) using `f_step`:

$$\begin{aligned}\texttt{f\_ifelse}[\texttt{cond},\texttt{t},\texttt{f},\lambda](\boldsymbol{x}) := {}&\texttt{ReLU(-}\lambda\texttt{+}\lambda\texttt{f\_step(}{}^1\!/\!_2\texttt{-cond(}\boldsymbol{x}\texttt{))+f(}\boldsymbol{x}\texttt{))} + \texttt{ReLU(-}\lambda\texttt{+}\lambda\texttt{f\_step(cond(}\boldsymbol{x}\texttt{)-}{}^1\!/\!_2\texttt{)+t(}\boldsymbol{x}\texttt{))}\\ &- \texttt{ReLU(-}\lambda\texttt{+}\lambda\texttt{f\_step(}{}^1\!/\!_2\texttt{-cond(}\boldsymbol{x}\texttt{))-f(}\boldsymbol{x}\texttt{))} - \texttt{ReLU(-}\lambda\texttt{+}\lambda\texttt{f\_step(cond(}\boldsymbol{x}\texttt{)-}{}^1\!/\!_2\texttt{)-t(}\boldsymbol{x}\texttt{))}.\end{aligned} \tag{9}$$

We can also apply the optimisation strategy here. While this implementation is robust to noise in the input it appears to be more sensitive to parameter noise, as shown in Fig. 4.

Numerically stable `f_ifelse` with multiplicative gates. Removing the unused `ReLU` terms in the original `f_ifelse` reduces the opportunities for numerical precision issues to creep in but does not solve the underlying problem. The multiplicative gating present in the Linear Gated RNN (Eq. (4)) and Gated RNN models (Eq. (5)) can help via implementing a numerically stable conditional operator:

$$\texttt{f\_ifelse}[\texttt{cond}, \texttt{t}, \texttt{f}](\boldsymbol{x}) := \texttt{cond}(\boldsymbol{x}) \odot \texttt{t}(\boldsymbol{x}) + \texttt{f\_not}(\texttt{cond}(\boldsymbol{x})) \odot \texttt{f}(\boldsymbol{x}), \tag{10}$$

where the element-wise product is implemented in LSRL with `Concat` and `Multi`. We will refer to the implementation of `f_ifelse` in Eq. (10) as `multiplicative`. Similarly to original implementation of `f_ifelse` in Eq. (7), we can drop the $\texttt{t}(\boldsymbol{x})$ and $\texttt{f}(\boldsymbol{x})$ term if they are equal to zero (`multiplicative optimized`). If $\texttt{cond}(\boldsymbol{x})$ is not exactly zero, $\texttt{cond}(\boldsymbol{x}) \odot \texttt{t}(\boldsymbol{x})$ will result in a small error to the output but, in contrast to the original implementation, is not going to cause a discontinuity in the output of the operation. Therefore, Eq. (10) should be more robust to numerical precision issues than Eq. (7). Fig. 4 shows that this is the case in practice with Lsts. 1 and 2 being more robust to parameter noise when using multiplicative gates compared to the `ReLU`-based implementations. Therefore, Linear Gated RNNs (Eq. (4)) —to which models with multiplicative gates can be compiled— are more likely than Linear RNNs (Eq. (3)) to exhibit universal approximation properties in practice.

6 Universal In-context Approximation with Non-linear (Gated) RNNs

Secs. 4 and 5 showed how universal approximation of continuous and token-to-token functions can be implemented in LSRL and compiled to respectively Linear RNNs and Linear Gated RNNs. This section aims to address the situation with *non-linear* state updates, that is, the cases of classic and gated RNNs (Eqs. (2) and (5)). Concretely, we show how every *linear* (Gated) RNN can be converted to a *non-linear* (Gated) RNN. The key idea is that the `ReLU` applied to the state updates in the non-linear architectures is an identity operation if its inputs are positive. Hence, we can split the states in positive and negative components, flip the sign of the negative component, pass them separately through the `ReLU`—which will act as an identity as all elements will be non-negative— and then fuse the positive and negative components back together in the $\boldsymbol{A}$ matrix at the next time step:

$$\begin{aligned} \boldsymbol{s}_t &= \boldsymbol{A}\boldsymbol{s}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{b} \\ \boldsymbol{y}_t &= \phi(\boldsymbol{s}_t). \end{aligned} \quad \equiv \quad \begin{aligned} \begin{bmatrix} \boldsymbol{s}_t^+ \\ \boldsymbol{s}_t^- \end{bmatrix} &= \texttt{ReLU}\left(\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{A} \\ -\boldsymbol{A} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_t^+ \\ \boldsymbol{s}_t^- \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ -\boldsymbol{B} \end{bmatrix} \boldsymbol{x}_t + \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \right) \\ \boldsymbol{y}_t &= \phi\left([\boldsymbol{I} \quad -\boldsymbol{I}] \begin{bmatrix} \boldsymbol{s}_t^+ \\ \boldsymbol{s}_t^- \end{bmatrix} \right). \end{aligned} \tag{11}$$

Using Eq. (11) we can compile any LSRL program to an RNN (Eq. (2)) or a Gated RNN (Eq. (5)). This includes Lsts. 1 and 2. Hence, RNNs and Gated RNNs can be universal in-context approximators for continuous and token-to-token functions. As any Gated RNN can be represented as a GRU model (App. C) or an LSTM (App. D), these models are too universal in-context approximators.

7 Discussion and Conclusions

We developed LSRL: a programming language for specifying programs expressible with recurrent neural architectures. We then used LSRL to show that various architectures —from the humble RNN to the state-of-the-art Linear Gated RNNs— can all be universal approximators *in-context*.

Safety and security implications. If a model can be prompted to approximate any function, then preventing it from exhibiting undesirable behaviours (i.e., alignment) might be impossible. Therefore, it is important to further study the safety and security implications of these properties.

Limitations. In this work we provide *constructive existence results*: that is, we show that there can exist models with various recurrent architectures that are universal in-context approximators. However, the present theory is not sufficient to analyse whether *a given model* has this property. That is a much more difficult question that would require a very different approach. We also assume no restrictions on the $\boldsymbol{A}$ matrix in the state update equations. However, many state-of-the-art models impose structural constraints on $\boldsymbol{A}$ (e.g., it being diagonal) for the sake of fast training and inference (Gu et al., 2020, 2021; Gupta et al., 2022). It is not directly obvious whether such structural restrictions would affect the universal in-context approximation abilities of these architectures. In practice, however, the compiled matrices are very sparse and often diagonal. Therefore, it is highly likely that our results translate to models with structural restrictions.

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A Computation Graph Debranching Rules

We convert the computation DAG resulting from the LSRL program into a path program by attending to the first node whose output is the input for multiple other nodes, i.e., the first branching node.

Preparation step. Before we even start debranching we first pre-process the graph by fusing consecutive nodes of the same type together. The specific rules are:

- If a `Lin` node is followed by a single other `Lin` node, then fuse them together. This follows directly from the classical result that composing linear functions is a linear function.
- If a `ReLU` node is followed by another `ReLU` node, we can drop one of them as `ReLU` is idempotent.
- If a `Lin` is followed by a `LinState`, we can subsume the weight matrix $\boldsymbol{A}$ of the linear node in the $\boldsymbol{B}$ matrix of the `LinState`, and the bias $\boldsymbol{b}$ of the `Lin` node in the bias $\boldsymbol{b}$ of the `LinState`.
- If all inputs of a `Concat` node are the same, then this node only duplicates the input and hence can be safely replaced with a `Lin` layer.

The debranching process goes through the following cases in order. And iterates until there are no branching nodes left, in other words, until the graph has become a path graph. We will refer to the nodes whose input is the branching node as *subsequent nodes*.

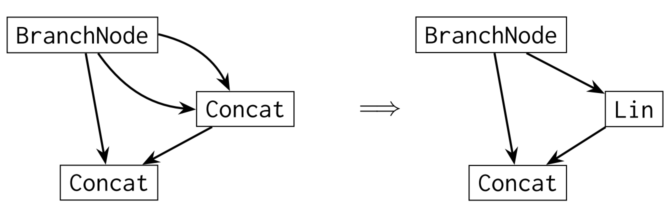
Case 1A: If all subsequent nodes are `Multi`. As all `Multi` nodes that have the same input (the branching node) they must all be producing the exact same output. Hence, only one can be kept. This removes one branch.

Case 1B: If subsequent nodes are a combination of `Multi` and other nodes. We add a single `Lin` layer that acts as a bypass for the non-`Multi` nodes using the fact that multiplicatin by 1 is identity. This is followed by a single `Multi` layer. We then add `Slice` operators between the new `Lin` layer and the non-`Multi` nodes. This keeps the number of branches unchanged but removes the `Multi` node and the new branch can be handled by the other rules.

Case 2: All subsequent nodes are `LinState`. `LinState` nodes can be fused into a single `LinState` node by combining their states and update matrices. As each `LinState` may have different subsequent nodes itself, we add `Slice` nodes to extract the respective subspaces of the state. This keeps the number of branches unchanged but puts the graph into Case 5A.

Case 3: All subsequent nodes are `ReLU`. We can replace them by a single `ReLU` node. This removes one branch.

Case 4: All subsequent nodes are `Concat`. One complication is that `Concat` nodes can depend on other `Concat` nodes. So, we will restrict ourselves at this step by only treating the `Concat` nodes that depend only on the branch node directly by replacing them with a single `Lin` node. The rest will be handled by the `Lin` and `Concat` case (Case 10) or the only `Lin` case (Cases 5A and 5B). See the following example:



Hence, this operation either reduces the number of branches by one or will be followed by a case that reduces the number of branches.

Case 5A: Only `Lin` nodes and they are all `Slices`. This is one of the more challenging cases. While the `Slice` nodes are simply `Lin` nodes with special structure, we cannot treat them like standard `Lin` nodes (see Case 5B). While we can merge them into a single `Lin` node, we will then need further `Slices` to extract the relevant subspaces for the subsequent nodes. Therefore, we would be simply replacing `Slice` nodes with `Slice` nodes. Instead, we use the observation that `Slice` nodes can be fused with subsequent `Lin` and `LinState` nodes and can be pushed after `ReLU` and `Concat` nodes. Therefore we treat each subsequent node differently, depending on its type:

- If there are `Multi` nodes after any of the `Slice` nodes, they can all be fused into a single `Lin` node followed by a single `Multi` node.
- If there are `Lin` or `LinState` nodes after any of the `Slice` nodes, the `Slices` can be fused with the $\boldsymbol{A}$ matrix of the `Lin` nodes and the $\boldsymbol{B}$ matrix of the `LinState` nodes. This uses the fact that composing linear functions results in a linear function.
- If there is a `ReLU` after a `Slice` node, their position can be switched without changing the nodes. That is because `ReLU` commutes with linear operations with $\boldsymbol{b} = \mathbf{0}$ and $\boldsymbol{A}$ with non-negative eigenvalues as is the case for `Slice` nodes.
- If there is a `Concat` node after a `Slice` node, we can similarly push the `Slice` as a new `Lin` node after the `Concat`.

This step does not reduce the number of branching nodes but prepares the graph for a removal, with the specific case depending on the remaining nodes.

Case 5B: Only `Lin` nodes and they are not all `Slices`. We can combine them into a single `Lin` node and then add `Slices` to extract the relevant subspaces for the subsequent nodes. These `Slices` can then be pushed into the next operations using Case 5A.

Case 6: Both `LinState` nodes and other nodes. If both `LinState` nodes and other nodes are present, we can pass through the other variables with dummy `LinState` variables using zero matrices for $\boldsymbol{A}$ and identities for $\boldsymbol{B}$. Then, Case 2 can be used to fuse all the `LinState` variables together.

Case 7A: Only `Lin` and `ReLU` nodes where all `Lin` nodes are followed by only one node which is a `ReLU`. If we add `Lin` bypasses to the `ReLU`s we will have only `Lin` nodes left. Each one of them would be followed by a `ReLU`. Hence, Case 5B can be first applied, followed by Case 3.

Case 7B: Only `Lin` and `ReLU` nodes where some `Lin` nodes are not followed by only one node which is a `ReLU`. In this case we cannot apply the above strategy. Instead, we fuse the `ReLU`s by placing `ReLU`-based bypasses before the `Lin` nodes. We do this in a similar spirit to Eq. (11), by splitting the positive and negative components and treating them separately. See App. F.6 for the LSRL implementation. Our DAG will then be in Case 7A first, then Case 5B, and, finally, in Case 3.

Case 8: Only `Lin` and `Concat` nodes. We add `Lin` bypasses for the `Concat` nodes which can then be merged using Case 5B and then Case 5A.

Case 9: Only `ReLU` and `Concat` nodes. Same strategy as for Case 8 but with `ReLU` bypasses.

Case 10: Only `Lin`, `ReLU` or `Concat` nodes. We introduce `ReLU` bypasses to all `Concat` nodes and to the `Lin` branches which are not immediately followed by a `ReLU`. This will be followed by applying Case 5B and then Case 3.

The above 13 cases cover all possible branching configurations. After repeated application, they reduce any DAG corresponding to an LSRL program to a path graph that can be compiled to one of the recurrent models in Sec. 2.

B Error Bound on the Approximation Scheme for Continuous Functions

In Sec. 4.1 we outlined a strategy to perform universal in-context approximation for continuous functions with Linear RNNs. The full program is in Lst. 1 and an illustration of the scheme is presented in Fig. 2. In Sec. 4.1 we claimed that the prompt length required to approximate an L-Lipschitz function f (w.r.t. the ℓ_2 norm) to precision ϵ is $N = (2\epsilon/L\sqrt{d_{\text{in}}})^{-d_{\text{in}}} = \mathcal{O}(\epsilon^{-d_{\text{in}}})$. The present appendix offers the formal proof of this claim.

The program in Lst. 1 approximates the value of a function $\boldsymbol{y} = f(\boldsymbol{q})$ with the value $\bar{\boldsymbol{y}}$ at the centre $\boldsymbol{c}$ of the cell that contains $\boldsymbol{q}$. Therefore, the error of our approximation is the maximum difference between $f(\boldsymbol{q})$ and $f(\boldsymbol{c})$: $\|f(\boldsymbol{q}) - f(\boldsymbol{c})\|_2$. First, as the length of each side of the cell is δ, that means that $\|\boldsymbol{q} - \boldsymbol{c}\|_\infty \leq \delta/2$. Thus, $\|\boldsymbol{q} - \boldsymbol{c}\|_2 \leq \sqrt{d_{\text{in}}}\delta/2$. Therefore, thanks to f being L-Lipschitz we get:

$$\|f(\boldsymbol{q}) - f(\boldsymbol{c})\|_2 \leq \frac{\delta L\sqrt{d_{\text{in}}}}{2}.$$

If we want to upper bound this approximation error by ϵ, we need to have δ small enough:

$$\delta \leq \frac{2\epsilon}{L\sqrt{d_{\text{in}}}}.$$

Finally, as the number of cells we need to cover the whole domain is $N = (1/\delta)^{d_{\text{in}}}$, this corresponds to us needing sufficiently long prompt:

$$N \geq \left(\frac{1}{\delta}\right)^{d_{\text{in}}} \geq \left(\frac{L\sqrt{d_{\text{in}}}}{2\epsilon}\right)^{d_{\text{in}}}.$$

Therefore, if we want our approximation to have error at most ϵ anywhere in the domain, we need a prompt of length at least $(L\sqrt{d_{\text{in}}}/2\epsilon)^{d_{\text{in}}}$.

C Gated RNNs are GRU models

A GRU layer (Cho et al., 2014) with input $\boldsymbol{a}_t \in \mathbb{R}^{d_{\texttt{in}}}$ and hidden state $\boldsymbol{h}_{t-1} \in \mathbb{R}^{d_{\texttt{hidden}}}$, and output $\boldsymbol{h}_t \in \mathbb{R}^{d_{\texttt{hidden}}}$ can be described as follows:

$$\boldsymbol{z}_t = \texttt{Sigmoid}(\boldsymbol{W}_z\boldsymbol{a}_t + \boldsymbol{U}_z\boldsymbol{h}_{t-1} + \boldsymbol{b}_z), \qquad \text{(update gate vector)} \tag{12}$$

$$\boldsymbol{r}_t = \texttt{Sigmoid}(\boldsymbol{W}_r\boldsymbol{a}_t + \boldsymbol{U}_r\boldsymbol{h}_{t-1} + \boldsymbol{b}_r), \qquad \text{(reset gate vector)} \tag{13}$$

$$\hat{\boldsymbol{h}}_t = \texttt{tanh}(\boldsymbol{W}_h\boldsymbol{a}_t + \boldsymbol{U}_h(\boldsymbol{r}_t \odot \boldsymbol{h}_{t-1}) + \boldsymbol{b}_h), \qquad \text{(candidate activation vector)} \tag{14}$$

$$\boldsymbol{h}_t = (1 - \boldsymbol{z}_t) \odot \boldsymbol{h}_{t-1} + \boldsymbol{z}_t \odot \hat{\boldsymbol{h}}_t, \qquad \text{(output vector)} \tag{15}$$

In this section, we show a conversion of a single Gated RNN layer (Eq. (5)) to $k+2$ GRU layers. Here, k is the number of layers in the γ and h MLPs in Eq. (5). We first show that a single GRU layer can be used to compute the updated state $\boldsymbol{s}_t$ and the output of the first layer of γ when applied to $\boldsymbol{x}_t$. Then, every pair of single layers of $\gamma(\boldsymbol{x}_t)$ and $\phi(\boldsymbol{s}_t)$ can be represented as an individual GRU layer. Finally, a single layer can be used to compute the element-wise multiplication $\gamma(\boldsymbol{x}_t) \odot \phi(\boldsymbol{s}_t)$. For simplicity, we assume the `Sigmoid` and `tanh` nonlinearities are replaced by `ReLU`s. If not, they can each be approximated with MLPs and hence also with additional GRU layers. Additionally, for convenience we will assume $d_{\texttt{in}} = d_{\texttt{hidden}}$.

C.1 Representing the state update as a GRU layer

For this layer we set $\boldsymbol{b}_z = \boldsymbol{1}$, $\boldsymbol{W}_z = \boldsymbol{0}$, $\boldsymbol{U}_z = \boldsymbol{0}$ giving $\boldsymbol{z}_t = \boldsymbol{1}$. Similarly, we set $\boldsymbol{b}_r = \boldsymbol{1}$, $\boldsymbol{W}_r = \boldsymbol{0}$, $\boldsymbol{U}_r = \boldsymbol{0}$ giving $\boldsymbol{r}_t = \boldsymbol{1}$. Thus, Eq. (14) reduces to:

$$\hat{\boldsymbol{h}}_t = \sigma(\boldsymbol{W}_h\boldsymbol{a}_t + \boldsymbol{U}_h\boldsymbol{h}_{t-1} + \boldsymbol{b}_h), \tag{16}$$

Setting $\boldsymbol{a}_t = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{x}_t\end{bmatrix}$, where $\boldsymbol{x}_t \in \mathbb{R}^{d_{\text{in}}/2}$, $\boldsymbol{h}_{t-1} = \begin{bmatrix}\boldsymbol{s}_{t-1}\\ \boldsymbol{0}\end{bmatrix}$, where $\boldsymbol{s}_{t-1} \in \mathbb{R}^{d_{\texttt{hidden}}/2}$, $\boldsymbol{W}_h = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{B}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}$, $\boldsymbol{U}_h = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}$, $\boldsymbol{b}_h = \begin{bmatrix}\boldsymbol{b}\\ -\boldsymbol{k}_{lb}\end{bmatrix}$, where $\boldsymbol{k}_{lb}$ is a vector where every element in $\boldsymbol{k}$ is a lower bound on $\boldsymbol{x}_t$. results in Eq. (15) becoming:

$$\boldsymbol{h}_t = \sigma\left(\begin{bmatrix}\boldsymbol{0} & \boldsymbol{B}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{x}_t\end{bmatrix} + \begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{s}_{t-1}\\ \boldsymbol{0}\end{bmatrix} + \begin{bmatrix}\boldsymbol{b}\\ -\boldsymbol{k}_{lb}\end{bmatrix}\right) = \begin{bmatrix}\sigma(\boldsymbol{A}\boldsymbol{s}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{b})\\ \sigma(\boldsymbol{x}_t - \boldsymbol{k}_{lb})\end{bmatrix} = \begin{bmatrix}\sigma(\boldsymbol{s}_t)\\ \boldsymbol{x}_t - \boldsymbol{k}_{lb}\end{bmatrix}. \tag{17}$$

Note: if we do not want to assume a compact domain for $\boldsymbol{x}_t$, it would be possible to use the same trick as in Equation (11) rather than subtracting $\boldsymbol{k}$ in this layer and adding in the next. However, we omit this approach for clarity of presentation.

C.2 Representing each MLP layer as a GRU layer

In these layers, similarly to the recurrent layer, we set $\boldsymbol{b}_z = \mathbf{1}$, $\boldsymbol{W}_z = \mathbf{0}$, $\boldsymbol{U}_z = \mathbf{0}$ giving $\boldsymbol{z}_t = \mathbf{1}$. In the same way, we set $\boldsymbol{b}_r = \mathbf{1}$, $\boldsymbol{W}_r = \mathbf{0}$, $\boldsymbol{U}_r = \mathbf{0}$ giving $\boldsymbol{r}_t = \mathbf{1}$. Here, however, we set $\boldsymbol{W}_h = \begin{bmatrix} \boldsymbol{W}_{h_i} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{W}_{\gamma_i} \end{bmatrix}$, $\boldsymbol{U}_h = \mathbf{0}$ and $\boldsymbol{b}_h = \begin{bmatrix} \boldsymbol{b}_{h_i} \\ \boldsymbol{b}_{\gamma_i} \end{bmatrix}$, except for the first of such layer where $\boldsymbol{b}_h = \begin{bmatrix} \boldsymbol{b}_{h_i} \\ \boldsymbol{b}_{\gamma_i} + \boldsymbol{W}_{\gamma_i} \boldsymbol{k}_{lb} \end{bmatrix}$. Thus, for an input $\boldsymbol{a}_t = \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix}$ the layer output (Eq. (15)) for layer i is:

$$\boldsymbol{h}_t = \sigma\left(\begin{bmatrix} \boldsymbol{W}_{\phi_i} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{W}_{\gamma_i} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_{\phi_i} \\ \boldsymbol{b}_{\gamma_i} \end{bmatrix}\right) = \begin{bmatrix} \phi_i(\boldsymbol{a}_{1,t}) \\ \gamma_i(\boldsymbol{a}_{1,t}) \end{bmatrix}. \tag{18}$$

Here, ϕ_i and γ_i are the i-th layers (including the `ReLU`) of respectively ϕ and γ in Eq. (5).

C.3 Representing the multiplicative gating with a single GRU layer

The only thing left is to model the element-wise multiplication of the outputs of ϕ and γ in Eq. (5). We do this using a GRU layer with $\boldsymbol{b}_z = \mathbf{0}$, $\boldsymbol{W}_z = \mathbf{0}$, $\boldsymbol{U}_z = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix}$. We set $\boldsymbol{b}_r = \mathbf{0}$, $\boldsymbol{W}_r = \mathbf{0}$, $\boldsymbol{U}_r = \mathbf{0}$ giving $\boldsymbol{r}_t = \mathbf{0}$. We also set $\boldsymbol{b}_h = \mathbf{0}$, $\boldsymbol{W}_h = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix}$, $\boldsymbol{U}_h = \mathbf{0}$. Thus, for an input $\boldsymbol{a}_t = \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix}$, the output $\boldsymbol{h}_t$ (Eq. (15)) of this GRU layer becomes:

$$\boldsymbol{h}_t = \sigma\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix}\right) \odot \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \sigma(\boldsymbol{a}_{1,t}) \odot \boldsymbol{a}_{2,t} \end{bmatrix}. \tag{19}$$

If $\boldsymbol{a}_t$ is the output of a GRU layer constructed as in Eq. (18) (as is in our case), then it must be non-negative. This is due to the `ReLU` application in Eq. (18). Hence, the application of another `ReLU` to $\boldsymbol{a}_{1,t}$ in Eq. (19) can be safely removed as `ReLU` is idempotent and Eq. (19) simplifies to

$$\boldsymbol{h}_t = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{a}_{1,t} \odot \boldsymbol{a}_{2,t} \end{bmatrix}. \tag{20}$$

Thus, this construction computes element-wise multiplication of $\boldsymbol{a}_{1,t}$ and $\boldsymbol{a}_{2,t}$.

C.4 Composing the operations to model a single Gated RNN layer

In order to represent Eq. (5), we use one GRU layer for the recurrence (as described in App. C.1), followed by k GRU layers modelling a pair of the k MLP layers of ϕ and γ (App. C.2), completed with a single mixing layer (App. C.3). This stack of $k + 2$ layers models exactly the Gated RNN layer (Eq. (5)):

$$\begin{aligned} \boldsymbol{s}_t &= \sigma\left(\boldsymbol{A} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{s}_{t-1} \end{bmatrix} + \boldsymbol{B} \begin{bmatrix} \boldsymbol{x}_t \\ \mathbf{0} \end{bmatrix} + \boldsymbol{b}\right) \\ \boldsymbol{y}_t &= \begin{bmatrix} \mathbf{0} \\ \gamma(\boldsymbol{x}_t) \odot \phi(\boldsymbol{s}_t) \end{bmatrix}, \end{aligned}$$

With this, we have shown that any Gated RNN (Eq. (5)) can be expressed as a GRU-based model. Hence, the two universal approximation programs in Lsts. 1 and 2 can be implemented also in GRU-based models. Thus, the GRU architecture can also be a universal in-context approximator.

D Gated RNNs are LSTMs

A single LSTM layer (Hochreiter and Schmidhuber, 1997; Gers et al., 2000) with input $\boldsymbol{a}_t \in \mathbb{R}^{d_{\text{in}}}$, hidden state $\boldsymbol{h}_{t-1} \in \mathbb{R}^{d_{\text{hidden}}}$, candidate memory cell $\tilde{\boldsymbol{c}}_t \in \mathbb{R}^{d_{\text{hidden}}}$, memory cell $\boldsymbol{c}_t \in \mathbb{R}^{d_{\text{hidden}}}$ and layer output $\boldsymbol{h}_t \in \mathbb{R}^{d_{\texttt{hidden}}}$ can be expressed as:

$$\boldsymbol{f}_t = \texttt{Sigmoid}(\boldsymbol{W}_f \boldsymbol{a}_t + \boldsymbol{U}_f \boldsymbol{h}_{t-1} + \boldsymbol{b}_f), \quad \text{(forget gate vector)} \tag{21}$$
$$\boldsymbol{i}_t = \texttt{Sigmoid}(\boldsymbol{W}_i \boldsymbol{a}_t + \boldsymbol{U}_i \boldsymbol{h}_{t-1} + \boldsymbol{b}_i), \quad \text{(input gate vector)} \tag{22}$$
$$\boldsymbol{o}_t = \texttt{Sigmoid}(\boldsymbol{W}_o \boldsymbol{a}_t + \boldsymbol{U}_o \boldsymbol{h}_{t-1} + \boldsymbol{b}_o), \quad \text{(output gate vector)} \tag{23}$$
$$\tilde{\boldsymbol{c}}_t = \texttt{tanh}(\boldsymbol{W}_c \boldsymbol{a}_t + \boldsymbol{U}_c \boldsymbol{h}_{t-1} + \boldsymbol{b}_c), \quad \text{(candidate cell vector)} \tag{24}$$
$$\boldsymbol{c}_t = \boldsymbol{f}_t \odot \boldsymbol{c}_{t-1} + \boldsymbol{i}_t \odot \tilde{\boldsymbol{c}}_t, \quad \text{(memory cell vector)} \tag{25}$$
$$\boldsymbol{h}_t = \boldsymbol{o}_t \odot \texttt{tanh}(\boldsymbol{c}_t), \quad \text{(output vector)} \tag{26}$$

where $\boldsymbol{h}_0 = \mathbf{0}$ and $\boldsymbol{c}_0 = \mathbf{0}$.

In a way analogous to App. C, we show that a single layer of a gated RNN (Eq. (5)) can be expressed using $k+2$ LSTM layers, where k is the maximum depth of either of the MLP networks ϕ or γ. We again follow the setup of replacing all `Sigmoid` and `tanh` activation functions with `ReLU` activations which we denote σ and we again assume that $d_{\texttt{in}} = d_{\texttt{hidden}}$. The set up follows the same structure as in App. C. First, we show that the non-linear state update computing $\boldsymbol{s}_t$ can be expressed as a single LSTM layer. We then show that we can represent the layers in MLP networks $\gamma(\boldsymbol{x}_t)$ and $\phi(\boldsymbol{s}_t)$ using single LSTM layers. Finally, a single layer can compute the Hadamard product between $\gamma(\boldsymbol{x}_t)$ and $\phi(\boldsymbol{s}_t)$. Therefore, any Gated RNN with `ReLU` activations can be expressed as a LSTM with `ReLU` activations.

For clarity of the exposition, we once again assume that our inputs belong to a compact domain $\mathcal{X}$ of real vectors. This implies that the set is bounded and, in particular, that we can find a vector $\boldsymbol{k}_{lb}$ such that $\boldsymbol{k}_{lb,i} \leq (\boldsymbol{x}_t)_i$ for $i \in [d_{\texttt{in}}]$ for all $\boldsymbol{x}_t \in \mathcal{X}$. In other words, we have $(\boldsymbol{x}_t - \boldsymbol{k}_{lb})_i \geq 0$ for for $i \in 1, \ldots, d_{\texttt{in}}$. We will make use of this fact several times when dealing with `ReLU` activations.

D.1 Representing the state update as an LSTM layer

We first represent the non-linear state update in Eq. (5) using a single layer of an LSTM. In particular, we set $\boldsymbol{W}_f = \mathbf{0}$, $\boldsymbol{U}_f = \mathbf{0}$ and $\boldsymbol{b}_f = \mathbf{0}$ so that $\boldsymbol{f}_t = \mathbf{0}$. We also set $\boldsymbol{W}_i = \mathbf{0}$, $\boldsymbol{U}_i = \mathbf{0}$, $\boldsymbol{b}_i = \mathbf{1}$ and $\boldsymbol{W}_c = \mathbf{0}$, $\boldsymbol{U}_c = \mathbf{0}$, $\boldsymbol{b}_c = \mathbf{1}$. This results in $\boldsymbol{i}_t = \mathbf{1}$ and $\tilde{\boldsymbol{c}}_t = \mathbf{1}$. We see from this that the LSTM layer with these weight settings reduces to

$$\boldsymbol{h}_t = \boldsymbol{o}_t = \sigma(\boldsymbol{W}_o \boldsymbol{a}_t + \boldsymbol{U}_o \boldsymbol{h}_{t-1} + \boldsymbol{b}_o). \tag{27}$$

We now set $\boldsymbol{a}_t = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{x}_t \end{bmatrix}$, where $\boldsymbol{x}_t \in \mathbb{R}^{d_{\texttt{in}}/2}$, $\boldsymbol{h}_{t-1} = \begin{bmatrix} \boldsymbol{s}_{t-1} \\ \mathbf{0} \end{bmatrix}$, where $\boldsymbol{s}_{t-1} \in \mathbb{R}^{d_{\texttt{hidden}}/2}$, $\boldsymbol{W}_o = \begin{bmatrix} \mathbf{0} & \boldsymbol{B} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix}$, $\boldsymbol{U}_o = \begin{bmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $\boldsymbol{b}_o = \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{k}_{lb} \end{bmatrix}$ so that

$$\boldsymbol{h}_t = \sigma\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{B} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{x}_t \end{bmatrix} + \begin{bmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_{t-1} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{k}_{lb} \end{bmatrix}\right) = \begin{bmatrix} \sigma(\boldsymbol{A}\boldsymbol{s}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{b}) \\ \sigma(\boldsymbol{x}_t - \boldsymbol{k}_{lb}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_t \\ \boldsymbol{x}_t - \boldsymbol{k}_{lb} \end{bmatrix}. \tag{28}$$

D.2 Representing each MLP layer as an LSTM layer

Now we want to use an LSTM layers to model the MLP layers of both γ and ϕ simultaneously. We set $\boldsymbol{W}_f = \mathbf{0}$, $\boldsymbol{U}_f = \mathbf{0}$, $\boldsymbol{b}_f = \mathbf{0}$ and $\boldsymbol{W}_i = \mathbf{0}$, $\boldsymbol{U}_i = \mathbf{0}$, $\boldsymbol{b}_i = \mathbf{1}$ and $\boldsymbol{W}_c = \mathbf{0}$, $\boldsymbol{U}_c = \mathbf{0}$, $\boldsymbol{b}_c = \mathbf{1}$ as before. We make a change for these LSTM layers by setting $\boldsymbol{W}_o = \begin{bmatrix} \boldsymbol{W}_{\phi_i} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{W}_{\gamma_i} \end{bmatrix}$, $\boldsymbol{U}_o = \mathbf{0}$ and $\boldsymbol{b}_o = \begin{bmatrix} \boldsymbol{b}_{\phi_i} \\ \boldsymbol{b}_{\gamma_i} \end{bmatrix}$, except for the first layer where $\boldsymbol{b}_\phi = \begin{bmatrix} \boldsymbol{b}_{\phi_1} \\ \boldsymbol{b}_{\gamma_1} + \boldsymbol{W}_{\gamma_1}\boldsymbol{k} \end{bmatrix}$. Thus, for an input $\boldsymbol{a}_t = \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix}$ the layer output is:

$$\boldsymbol{h}_t = \sigma\left(\begin{bmatrix} \boldsymbol{W}_{\phi_i} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{W}_{\gamma_i} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_{\phi_i} \\ \boldsymbol{b}_{\gamma_i} \end{bmatrix}\right) = \begin{bmatrix} \phi_i(\boldsymbol{a}_{1,t}) \\ \gamma_i(\boldsymbol{a}_{2,t}) \end{bmatrix}. \tag{29}$$

Here, ϕ_i and γ_i again refer to the i-th layers (including the `ReLU`) of respectively ϕ and γ in Eq. (5).

Note that, without a loss of generality, if we have that ϕ has m layers whereas γ has k with $m < k$, then we can also model this by simply adding additional layers to model additional layers for γ whilst simply passing on ϕ unchanged. Specifically, we set set the weights to ensure that $\boldsymbol{f}_t = 0$ and that $\boldsymbol{i}_t$ and $\tilde{\boldsymbol{c}}_t$ are $\mathbf{1}$ so that $\boldsymbol{h}_t = \boldsymbol{o}_t$. The input to this layer for $i > k$ is then given as $\boldsymbol{a}_t = \begin{bmatrix} \phi(\boldsymbol{s}_t) \\ \boldsymbol{a}_{2,t} \end{bmatrix}$. The we set the weights to compute $\boldsymbol{o}_t$ as

$$\boldsymbol{o}_t = \sigma\left(\begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{W}_{\gamma_i} \end{bmatrix} \begin{bmatrix} \phi(\boldsymbol{s}_t) \\ \boldsymbol{a}_{2,t} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{b}_{\phi_i} \end{bmatrix}\right) = \begin{bmatrix} \phi(\boldsymbol{s}_t) \\ \gamma_i(\boldsymbol{a}_{2,t}) \end{bmatrix}. \tag{30}$$

D.3 Representing the multiplicative gating with an LSTM layer

Finally, we model the element-wise multiplication of the outputs of ϕ and γ in Eq. (5). To do this we set the weights of the input gate and candidate cell vectors for the final layers of of γ and ϕ to be as follows:

$$\boldsymbol{i}_t = \sigma\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{a}_{1,t} \end{bmatrix} \tag{31}$$

and

$$\tilde{\boldsymbol{c}} = \sigma\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{a}_{2,t} \end{bmatrix}. \tag{32}$$

Then by setting $\boldsymbol{W}_f = \mathbf{0}$, $\boldsymbol{U}_f = \mathbf{0}$, $\boldsymbol{b}_f = \mathbf{0}$ and $\boldsymbol{W}_o = \mathbf{0}$, $\boldsymbol{U}_o = \mathbf{0}$, $\boldsymbol{b}_o = \mathbf{1}$ to force $\boldsymbol{f}_t = \mathbf{0}$ and $\boldsymbol{o}_t = \mathbf{1}$, we get

$$\boldsymbol{y}_t = \sigma(\boldsymbol{c}_t) = \begin{bmatrix} \sigma(\mathbf{0} \odot \mathbf{0}) \\ \sigma(\boldsymbol{a}_{1,t} \odot \boldsymbol{a}_{2,t}) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \sigma(\boldsymbol{a}_{1,t} \odot \boldsymbol{a}_{2,t}) \end{bmatrix}. \tag{33}$$

D.4 Composing the operations to model a single Gated RNN layer

To model the gated RNN described in Eq. (5), we again follow the same lines as described in App. C. In particular, we use one LSTM layer for the recurrent state updated as described in App. D.1. We then stack k LSTM layers as described in App. D.2 to model the k MLP layers of ϕ and γ in parallel. We then use one final layer to both give the final MLP layer of ϕ and γ and to compute their Hadamard product as set out in App. D.3 in order to match the output of the gated RNN in Eq. (5). Now, since we are working with $\sigma = \text{ReLU}$, both $\gamma(\boldsymbol{x}_t)$ and $\phi(\boldsymbol{s}_t)$ are positive and therefore so is their product. Hence, applying σ to the product components in Eq. (33) leaves the the components invariant. Therefore, we output is

$$\boldsymbol{y}_t = \begin{bmatrix} \mathbf{0} \\ \gamma(\boldsymbol{x}_t) \odot \phi(\boldsymbol{s}_t) \end{bmatrix}, \tag{34}$$

as required.

Hence, we have shown that a single layer of a gated RNN as described by Eq. (5) can be represented using $k + 2$ LSTM layers where k is the maximum depth of ϕ and γ. Therefore, once again, the two universal approximation programs in Lsts. 1 and 2 can also be implemented for LSTMs. Hence, LSTM models are also universal approximators in the sense described in Sec. 4.

E Gated Linear RNNs are Hawk/Griffin Models

A single residual block of a Hawk/Griffin model (De et al., 2024) consists of two components, a recurrent block for temporal mixing which makes use of a one-dimensional temporal convolution, as well as real-gated linear recurrent unit (RG-LRU) and a gated MLP block. Specifically, we consider an input $\boldsymbol{a}_t \in \mathbb{R}^{d_{\mathtt{in}}}$, inputs to the blocks of dimensions $d_{\mathtt{in}}$ and outputs from each block of dimensions $d_{\mathtt{in}}$. Within blocks, all vectors have dimensionality $d_{\mathtt{hidden}} = Ed_{\mathtt{in}}$, where E is denotes an expansion factor. Below, we formally describe the form of the recurrent and gated MLP blocks which are the two main components making up the residual blocks used for Hawk and Griffin.

Recurrent block. The recurrent block consists of two branches. The first applies a one-dimensional temporal convolution followed by a RG-LRU. The second branch simply performs a linear transformation followed by a non-linearity, i.e. applies a single layer of an MLP.

Consider the first branch of the recurrent block with an input $\boldsymbol{a}_t$. The one-dimensional temporal convolution can be written as:

$$\boldsymbol{a}'_t = \boldsymbol{W}_a \boldsymbol{a}_t, \tag{35}$$

$$\boldsymbol{g}_t = \texttt{GeLU}(\boldsymbol{W}_g \boldsymbol{a}_t + \boldsymbol{b}_g), \tag{36}$$

$$\boldsymbol{M}_t = \left[\boldsymbol{a}'_{t-(d_{\text{conv}}-1)}, \ldots, \boldsymbol{a}'_{t-2}, \boldsymbol{a}'_{t-1}, \boldsymbol{a}'_t\right], \tag{37}$$

$$\boldsymbol{z}_t = \sum_{i=0}^{d_{\text{conv}}-1} \boldsymbol{W}_M[i]\boldsymbol{M}_t[t-i] \; + \boldsymbol{b}_{\text{conv}} \qquad \text{(convolution with window size } d_{\text{conv}}\text{)}, \tag{38}$$

where $\boldsymbol{b}_{\text{conv}}$ is a bias vector and $\boldsymbol{W}_M = \left[\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{A}}^2\tilde{\boldsymbol{B}}, \cdots, \tilde{\boldsymbol{A}}^t\tilde{\boldsymbol{B}}, \cdots\right]$ is the convolutional kernel for the one-dimensional temporal convolution.

The output of this convolution is then fed into a RG-LRU. We can write this down concretely using as an input $\boldsymbol{z}_t$ from the one-dimensional convolution and with recurrent state $\boldsymbol{h}_t \in \mathbb{R}^{d_{\texttt{model}}}$:

$$\boldsymbol{r}_t = \texttt{Sigmoid}(\boldsymbol{W}_r \boldsymbol{z}_t + \boldsymbol{b}_r), \tag{39}$$

$$\boldsymbol{i}_t = \texttt{Sigmoid}(\boldsymbol{W}_i \boldsymbol{z}_t + \boldsymbol{b}_i), \tag{40}$$

$$a = \texttt{Sigmoid}(\Lambda), \qquad (\Lambda \text{ a learnable parameter}) \tag{41}$$

$$\boldsymbol{a}_t = a^{c\boldsymbol{r}_t}, \qquad (c = 8 \text{ fixed scalar constant}) \tag{42}$$

$$\boldsymbol{h}_t = \boldsymbol{a}_t \odot \boldsymbol{h}_{t-1} + \sqrt{1 - \boldsymbol{a}_t^2} \odot (\boldsymbol{i}_t \odot \boldsymbol{z}_t). \tag{43}$$

Now consider the second branch of the recurrent block. This performs a linear transformation followed by a non-linear activation:

$$\boldsymbol{g}_t = \texttt{GeLU}(\boldsymbol{W}_g \boldsymbol{a}_t + \boldsymbol{b}_g). \tag{44}$$

To get the final output of the recurrent block, we multiply the components of the vectors computed from each branch within the recurrent block and then perform a non-linear transformation:

$$\boldsymbol{h}'_t = \boldsymbol{g}_t \odot \boldsymbol{h}_t, \tag{45}$$

$$\boldsymbol{o}_t = \boldsymbol{W}_o \boldsymbol{h}'_t + \boldsymbol{b}_o. \tag{46}$$

Gated MLP block. After passing through the recurrent block, we pass the output $\boldsymbol{o}_t$ into a gated MLP block. Again we have two branches, the first where we linearly transform the input to this block

$$\boldsymbol{e}_t = \boldsymbol{W}_e \boldsymbol{o}_t + \boldsymbol{b}_e, \tag{47}$$

and the second performs a single layer MLP transformation as

$$\boldsymbol{f}_t = \texttt{GeLU}(\boldsymbol{W}_f \boldsymbol{o}_t + \boldsymbol{b}_f). \tag{48}$$

These are then combined through a Hadamard product and linear transformation as

$$\boldsymbol{e}'_t = \boldsymbol{e}_t \odot \boldsymbol{f}_t, \tag{49}$$

$$\boldsymbol{m}_t = \boldsymbol{W}_m \boldsymbol{e}'_t + \boldsymbol{b}_m. \tag{50}$$

We then have that the vector $\boldsymbol{m}_t$ acts as the output of the residual block given the input $\boldsymbol{a}_t$.

Distinction between the Griffin and Hawk models. Hawk is the more simple of the two architectures proposed in (De et al., 2024). Here, residual blocks using the recurrent block described above are simply stacked on top of each other to form the Hawk architecture. Griffin, on the other hand, mixes recurrent blocks and local attention. In particular, two residual blocks with recurrent blocks are followed by one residual block using local MQA attention (Beltagy et al., 2020; Shazeer, 2019).

Simplifying Assumptions. We again follow the setup of replacing all `Sigmoid` and `tanh` activation functions with `ReLU` activations which we denote σ. Furthermore, we assume for simplicity that $d_{\texttt{in}} = d_{\texttt{hidden}}$ by choosing $E = 1$. Moreover, the Hawk and Griffin architecture contains residual connections and normalising layers which we omit.[4] We again assume compactness of the input domain $\mathcal{X}$ and denote a vector of finite values $\boldsymbol{k}_{lb}$, such that $\boldsymbol{k}_{lb,i} \leq (\boldsymbol{x}_t)_i$ for $i \in [d_{\texttt{in}}]$ and all $\boldsymbol{x}_t \in \mathcal{X}$, just as before. Finally, we assume that $d_{\text{conv}} = T$ where T is the maximum sequence length.

[4]We will force a lot of our recurrent blocks to implement the identity function. So instead of this, we could implement the 0 function in the recurrent block and use a residual connection between the residual block input and the output of the recurrent block to achieve the same identity function. However, for clarity we ignore residual connections in our derivations.

E.1 Representing the state update using a recurrent block

Starting with the input to the Hawk model, which we denote $\boldsymbol{a}_t$, we define this to be a function of the input to the Gated RNN $\boldsymbol{x}_t$ as $\boldsymbol{a}_t = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{x}_t \end{bmatrix}$. First, we set $\boldsymbol{W}_a = \boldsymbol{I}$ so that $\boldsymbol{a}'_t = \boldsymbol{a}_t$. Next we choose matrices $\tilde{\boldsymbol{A}} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\tilde{\boldsymbol{B}} = \begin{bmatrix} \mathbf{0} & \boldsymbol{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ which we then use, with a convolutional window size of $d_{\text{conv}} = T$ to form the convolutional kernel $\boldsymbol{W}_M = \left[\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{A}}^2\tilde{\boldsymbol{B}}, \cdots, \tilde{\boldsymbol{A}}^t\tilde{\boldsymbol{B}}, \cdots\right]$. Setting the convolutional bias as $\boldsymbol{b}_{\text{conv}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ gives

$$\boldsymbol{z}_t = \sum_{i=0}^{t-1} \boldsymbol{W}_M[i]\boldsymbol{M}_t[t-i] \quad + \boldsymbol{b}_{\text{conv}}, \tag{51}$$

$$= \tilde{\boldsymbol{B}}\boldsymbol{a}_t + \tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}}\boldsymbol{a}_{t-1} + \cdots + \tilde{\boldsymbol{A}}^{t-1}\tilde{\boldsymbol{B}}\boldsymbol{a}_1 + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \tag{52}$$

$$= \begin{bmatrix} \boldsymbol{s}_t \\ \mathbf{1} \end{bmatrix}. \tag{53}$$

Now, we pass $\boldsymbol{z}_t$ through the RG-LRU. We set $\Lambda = 0$ so that $\boldsymbol{a}_t = 0$. We also define $\boldsymbol{W}_i = \mathbf{0}$ and $\boldsymbol{b}_i = \mathbf{1}$ so that $\boldsymbol{i}_t = \mathbf{1}$. This gives us $\boldsymbol{h}_t = \boldsymbol{z}_t$, so that we pass the output of the one-dimensional convolution through he RG-LRU.

Next, let's focus on the second branch. Making use of the lower bound $\boldsymbol{k}_{lb}$ on the domain $\mathcal{X}$, we set $\boldsymbol{W}_g = \boldsymbol{I}$ and $\boldsymbol{b}_g = \begin{bmatrix} \mathbf{1} \\ -\boldsymbol{k}_{lb} \end{bmatrix}$ so that

$$\boldsymbol{g}_t = \sigma\left(\boldsymbol{I}\begin{bmatrix} \mathbf{0} \\ \boldsymbol{x}_t \end{bmatrix} + \begin{bmatrix} \mathbf{1} \\ -\boldsymbol{k}_{lb} \end{bmatrix}\right) = \begin{bmatrix} \sigma(\mathbf{1}) \\ \sigma(\boldsymbol{x}_t - \boldsymbol{k}_{lb}) \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{x}_t - \boldsymbol{k}_{lb} \end{bmatrix}, \tag{54}$$

where we used that $(\boldsymbol{x}_t - \boldsymbol{k}_{lb})_i \geq \mathbf{0}$ for every i. Combining the two branches gives

$$\boldsymbol{h}'_t = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{x}_t - \boldsymbol{k}_{lb} \end{bmatrix} \odot \begin{bmatrix} \boldsymbol{s}_t \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_t \\ \boldsymbol{x}_t - \boldsymbol{k}_{lb} \end{bmatrix}. \tag{55}$$

We finally get the output of the recurrent block by defining $\boldsymbol{W}_o = \boldsymbol{I}$ and $\boldsymbol{b}_0 = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{k}_{lb} \end{bmatrix}$ so that

$$\boldsymbol{o}_t = \begin{bmatrix} \boldsymbol{s}_t \\ \boldsymbol{x}_t \end{bmatrix}. \tag{56}$$

E.2 Representing the identity function using a recurrent block

We now show that we can pass an input unchanged through a recurrent block. Assume that the input to the recurrent block is $\boldsymbol{a}_t = \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix}$ with $\boldsymbol{W}_a = \boldsymbol{I}$ so that $\boldsymbol{a}'_t = \boldsymbol{a}_t$. Then we define matrices $\tilde{\boldsymbol{A}} = \mathbf{0}$ and $\tilde{\boldsymbol{B}} = \boldsymbol{I}$ which we then use to form the convolutional kernel $\boldsymbol{W}_M = \left[\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{A}}^2\tilde{\boldsymbol{B}}, \cdots, \tilde{\boldsymbol{A}}^t\tilde{\boldsymbol{B}}, \cdots\right]$. Finally, setting the convolutional bias as $\boldsymbol{b}_{\text{conv}} = \mathbf{0}$ results in $\boldsymbol{z}_t = \boldsymbol{a}_t$. From here, we can again set $\Lambda = 0$, $\boldsymbol{W}_i = \mathbf{0}$ and $\boldsymbol{b}_i = \mathbf{1}$ so that $\boldsymbol{h}_t = \boldsymbol{z}_t$. Looking at the second branch and setting $\boldsymbol{W}_g = 0$ and $\boldsymbol{b}_g = \mathbf{1}$ so that $\boldsymbol{h}'_t = \boldsymbol{h}_t$. Finally, we can simply output the input to the recurrent block by setting $\boldsymbol{W}_o = \boldsymbol{I}$ and $\boldsymbol{b}_o = \mathbf{0}$ so that $\boldsymbol{o}_t = \boldsymbol{h}_t$ which means that $\boldsymbol{o}_t = \boldsymbol{a}_t$.

E.3 Representing each MLP layer as a gated MLP block

We can represent the MLP layers of the networks $\phi(\boldsymbol{s}_t)$ and $\gamma(\boldsymbol{x}_t)$ as described in Eq. (4) using Gated MLP blocks. We again denote the i-th layer of ϕ and γ as ϕ_i and γ_i. Assume that the input to the gated MLP block is $\boldsymbol{a}_t = \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix}$. Then, on the first purely linear branch, let us define $\boldsymbol{W}_e = \boldsymbol{I}$ and

$\boldsymbol{b}_e = \mathbf{1}$ so that $\boldsymbol{e}_t = \mathbf{1}$. On the second non-linear branch, we can define $\boldsymbol{W}_f = \begin{bmatrix} \boldsymbol{W}_{\phi_i} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{W}_{\gamma_i} \end{bmatrix}$ and $\boldsymbol{b}_f = \begin{bmatrix} \boldsymbol{b}_{\phi_i} \\ \boldsymbol{b}_{\gamma_i} \end{bmatrix}$. This results in

$$\boldsymbol{f}_t = \sigma\left(\begin{bmatrix} \boldsymbol{W}_{\phi_i} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{W}_{\gamma_i} \end{bmatrix}\begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_{\phi_i} \\ \boldsymbol{b}_{\gamma_i} \end{bmatrix}\right) = \begin{bmatrix} \phi_i(\boldsymbol{a}_{1,t}) \\ \gamma_i(\boldsymbol{a}_{2,t}) \end{bmatrix}. \tag{57}$$

Due to our setting of $\boldsymbol{e}_t$, we get $\boldsymbol{e}'_t = \boldsymbol{f}_t$. Further, defining $\boldsymbol{W}_m = \boldsymbol{I}$ and $\boldsymbol{b}_m = \mathbf{0}$ makes the output of the MLP block be

$$\boldsymbol{m}_t = \begin{bmatrix} \phi_i(\boldsymbol{a}_{1,t}) \\ \gamma_i(\boldsymbol{a}_{2,t}) \end{bmatrix}. \tag{58}$$

Emulating the layers of only one the two networks. Suppose without loss of generality (WLOG) that ϕ has m layers and γ has n layers where $m < n$. Suppose also that our input to the MLP block is $\boldsymbol{a}_t = \begin{bmatrix} \phi(\boldsymbol{x}_t) \\ \boldsymbol{a}_{2,t} \end{bmatrix}$. Again, on the first purely linear branch, let us define $\boldsymbol{W}_e = \boldsymbol{I}$ and $\boldsymbol{b}_e = \mathbf{1}$ so that $\boldsymbol{e}_t = \mathbf{1}$. Now we modify the weights on the second non-linear branch by defining $\boldsymbol{W}_f = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{W}_{\gamma_i} \end{bmatrix}$ and $\boldsymbol{b}_f = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{b}_{\gamma_i} \end{bmatrix}$. This gives us

$$\boldsymbol{f}_t = \sigma\left(\begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{W}_{\gamma_i} \end{bmatrix}\begin{bmatrix} \phi(\boldsymbol{x}_t) \\ \boldsymbol{a}_{2,t} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{b}_{\gamma_i} \end{bmatrix}\right) = \begin{bmatrix} \sigma(\phi(\boldsymbol{x}_t)) \\ \gamma_i(\boldsymbol{a}_{2,t}) \end{bmatrix} = \begin{bmatrix} \phi(\boldsymbol{x}_t) \\ \gamma_i(\boldsymbol{a}_{2,t}) \end{bmatrix}, \tag{59}$$

where we have used that since $\phi(\boldsymbol{x}_t)$ is a ReLU network whose final activation is a ReLU, we have that $\phi(\boldsymbol{x}_t) = \sigma(\phi(\boldsymbol{x}_t))$. Hence, if our networks have different depths and we have fully emulated one of the networks, we can continue to emulate the remaining layers of the other network while keeping the fully emulated network fixed and unchanged.

E.4 Representing the identify function using a gated MLP block

In this section we show that we can represent an identity function using a gated MLP block. This can be simply done by setting $\boldsymbol{W}_f = \mathbf{0}, \boldsymbol{b}_f = \mathbf{1}, \boldsymbol{W}_e = \boldsymbol{I}, \boldsymbol{b}_e = \mathbf{0}, \boldsymbol{W}_m = \boldsymbol{I}$ and $\boldsymbol{b}_m = \mathbf{0}$. This then gives us that for an input $\boldsymbol{a}_t = \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix}$ to the gated MLP block, the output of the gated MLP block is $\boldsymbol{m}_t = \boldsymbol{a}_t$. Thus, we pass the input through the gated MLP unchanged.

E.5 Representing multiplicative gating with a gated MLP block

The final thing we need to do is to compute an element-wise product of two vectors in order to match the output in Eq. (4). In other words, to match the $\phi(\boldsymbol{x}_t) \odot \gamma(\boldsymbol{s}_t)$ operation.

Again, assume that the input to the gated MLP block is $\boldsymbol{a}_t = \begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix}$. Working with the first linear branch, we define $\boldsymbol{W}_e = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix}$ and $\boldsymbol{b}_e = \mathbf{0}$, so that

$$\boldsymbol{e}_t = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{a}_{1,t} \\ \boldsymbol{a}_{2,t} \end{bmatrix} + \mathbf{0} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{a}_{1,t} \end{bmatrix}. \tag{60}$$

Next, we define $\boldsymbol{W}_f = \boldsymbol{I}$ and $\boldsymbol{b}_e = \mathbf{0}$ so that

$$\boldsymbol{f}_t = \begin{bmatrix} \sigma(\boldsymbol{a}_{1,t}) \\ \sigma(\boldsymbol{a}_{2,t}) \end{bmatrix}. \tag{61}$$

Setting $\boldsymbol{W}_m = \boldsymbol{I}$ and $\boldsymbol{b}_m = \mathbf{0}$ gives the output of the gated MLP as

$$\boldsymbol{m}_t = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{a}_{1,t} \odot \sigma(\boldsymbol{a}_{2,t}) \end{bmatrix}. \tag{62}$$

E.6 Composing the operations to model a single gated linear-RNN layer

Now that we have all the individual layers, we can combine them so that we can use a Hawk model to emulate a single Gated RNN layer.

First we start by taking the input of the form $\boldsymbol{a}_t = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{x}_t \end{bmatrix}$. We use a residual block that consists of a recurrent block computing the state update as descried in App. E.1 and then a gated MLP block that computes the identity function as demonstrated in App. E.4. This gives an output from this first recurrent block as $\boldsymbol{o}_t = \begin{bmatrix} \boldsymbol{s}_t \\ \boldsymbol{x}_t \end{bmatrix}$.

Next, we emulate the MLP layers of the networks ϕ and γ in parallel. Suppose WLOG that ϕ and γ have m and n MLP layers respectively, where $m \leq n$. We stack m residual blocks using recurrent blocks that implement the identity function as described in App. E.2 followed by MLP blocks that apply the MLP layers of ϕ and γ as described in App. E.3. Stacking m such residual blocks results in the output $\boldsymbol{m}_t = \begin{bmatrix} \gamma_m(\boldsymbol{s}_t) \\ \phi(\boldsymbol{x}_t) \end{bmatrix}$, where we can fully emulate the shallower network $\phi(\boldsymbol{x}_t)$.

Now, for the remaining $k - m$ layers for the network $\gamma(\boldsymbol{x}_t)$, we stack residual blocks with recurrent blocks implementing the identity function as described in App. E.2 and MLP blocks that leave $\phi(\boldsymbol{x}_t)$ unchanged whilst applying the additional layers needed to emulate $\gamma(\boldsymbol{s}_t)$ as described at the end of App. E.3. After stacking $k - m$ additional residual layers in this fashion, the output of the final residual block will now be $\boldsymbol{m}_t = \begin{bmatrix} \gamma(\boldsymbol{s}_t) \\ \phi(\boldsymbol{x}_t) \end{bmatrix}$, which fully reconstructs the MLP networks γ and ϕ.

Finally, we utilise a residual block with a recurrent block that implements the identity function as described in App. E.2 followed by a gated MLP block that applies multiplicative gating as described in App. E.5. This then gives as an output of this final residual block $\boldsymbol{m}_t = \begin{bmatrix} \mathbf{0} \\ \gamma(\boldsymbol{s}_t) \odot \sigma(\phi(\boldsymbol{x}_t)) \end{bmatrix}$. Since $\phi(\boldsymbol{x}_t)$ is a MLP network with the final activation function being a ReLU activation, we have that $\sigma(\phi(\boldsymbol{x}_t)) = \phi(\boldsymbol{x}_t)$, giving the required final output from the stacked block of residual blocks as

$$\boldsymbol{m}_t = \begin{bmatrix} \mathbf{0} \\ \gamma(\boldsymbol{s}_t) \odot \phi(\boldsymbol{x}_t) \end{bmatrix}. \tag{63}$$

Hence, we have shown that a single layer of a gated RNN as described by Eq. (5) can be represented using $k + 2$ Hawk residual blocks where k is the maximum depth of ϕ and γ. Once again, the two universal approximation programs in Lsts. 1 and 2 can also be applied to Hawk models as they can represent Gated Linear RNNs. Therefore, Hawk models are also universal approximators in the sense described in Sec. 4.

Gated Linear-RNNs are Griffin models too. The above argument extends to the Griffin architecture which uses stacks of two residual blocks with recurrent blocks followed by a residual block with attention. The only thing that changes is that for every third residual block, which in our argument will be used to compute the MLP layers of ϕ and γ in parallel, the recurrent block is now replaced with a local MQA block.

We can set the key query and values matrices to implement the identity function which is to act input to the block. Hence, as a corollary of the above argument, we can also show that the universal approximation programs in Lsts. 1 and 2 can also be implemented as Griffin models. Therefore, Griffin models can also be universal approximators in the sense described in Sec. 4.

F Definitions for some helper functions in LSRL

F.1 `f_not`

This is a convenience function that creates a NOT function block. It assumes that x is 0 or 1. Works with scalar and vector-valued inputs. With vector-valued inputs, it acts element-wise.

```
not_x = 1 - x
```

F.2 `f_and`

This is a convenience function that creates an AND function block. It assumes that x and y are 0 or 1. Works with scalar and vector-valued inputs. With vector-valued inputs, it acts element-wise. `mu` is the approximation parameter μ for `f_step` as described in Sec. 3.

```
and_x_y = ReLU(f_step(x, mu) + f_step(y, mu) - 1)
```

F.3 `f_or`

This is a convenience function that creates an OR function block. It assumes that x and y are 0 or 1. Works with scalar and vector-valued inputs. With vector-valued inputs, it acts element-wise. `mu` is the approximation parameter μ for `f_step` as described in Sec. 3.

```
or_x_y = f_step(x + y, mu=mu)
```

F.4 `f_smaller`

This is a convenience function that a less than comparison block. Works with scalar and vector-valued inputs. With vector-valued inputs, it acts element-wise. `mu` is the approximation parameter μ for `f_step` as described in Sec. 3.

```
smaller_x_y = f_step(y - x, mu=mu)
```

F.5 `f_larger`

This is a convenience function that a more than comparison block. Works with scalar and vector-valued inputs. With vector-valued inputs, it acts element-wise. `mu` is the approximation parameter μ for `f_step` as described in Sec. 3.

```
larger_x_y = f_step(x - y, mu=mu)
```

F.6 `f_relu_identity`

Identity operation using `ReLU`s. This is useful for debranching when some of the branches have `ReLU`s but the other don't. We can add this as a bypass for the ones that do not and can then merge the `ReLU`s together (see App. A for details).

```
positive_part = ReLU(x)
negative_part = ReLU(
    Linear(
        input=x,
        A=-1 * eye(x.dim),
        b=zeros(x.dim, 1),
    )
)
both = Concat([positive_part, negative_part])
relu_identity = Linear(
    input=both,
    A=hstack(eye(x.dim), -1 * eye(x.dim)),
    b=zeros(x.dim, 1),
)
```

F.7 `f_modulo_counter`

Computes the x `mod divisor` where x is a counter starting from zero. The idea is that we rotate a unit vector so that it makes a full revolution every `divisor` rotations. `dummy_input` can be any variable, we use it only to construct a constant.

```
angle = 2 * pi / divisor
R = [[cos(angle), sin(angle)], [sin(angle), cos(angle)]]
unit_vector = [[1], [0]]
# we first rotate, then output so if we want the first output to be 0 we need to have the init_state one step
     before that
init_state = R.inv() @ unit_vector
```

```
# this rotates a 2D vector 1/divisor revolutions at a time
cycler = LinState(
    input=dummy_input,
    A=R,
    B=zeros(2, dummy_input.dim),
    init_state=init_state,
)
# we now need to extract the position of the cycler
extractor_matrix = vstack(*[(R^i * unit_vector).T) for i in range(divisor)])
indicator = Linear(
    input=cycler,
    A=extractor_matrix,
    b=zeros(divisor, 1)
)
# the dot product with the row of extractor_matrix corresponding to the current position of the cycler is 1
# the dot product with the second highest is cos(angle)
# thus, we can threshold at 1-cos(angle/2) to get a one hot encoding of the current position of the cycler
one_hot = f_larger(indicator, cos(angle / 2))
# and to get an integer value we need one final linear layer
mod_value = Linear(
    one_hot,
    A=[[i for i in range(divisor)]],
    b=zeros(1, 1)
)
```

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